Chapter Five:
Nondeterministic Finite Automata

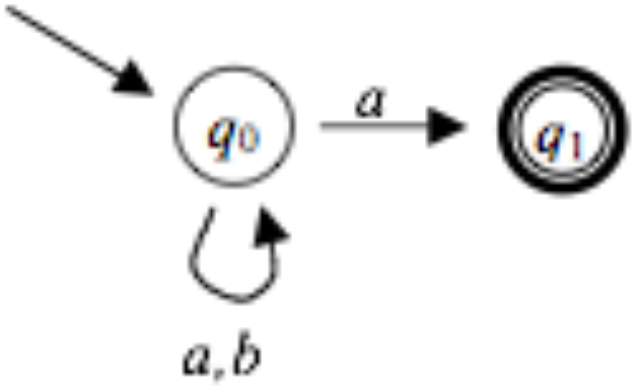
A DFA has exactly one transition from every state on every symbol in the alphabet. By relaxing this requirement we get a related but more flexible kind of automaton: the nondeterministic finite automaton or NFA.

NFAs are a bit harder to think about than DFAs, because they do not appear to define simple computational processes. They may seem at first to be unnatural, like puzzles invented by professors for the torment of students. But have patience! NFAs and other kinds of nondeterministic automata arise naturally in many ways, as you will see later in this book, and they too have a variety of practical applications.

Outline

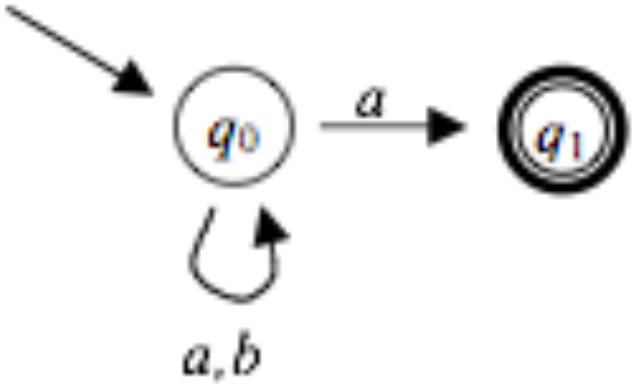
- 5.1 Relaxing a Requirement
- 5.2 Spontaneous Transitions
- 5.3 Nondeterminism
- 5.4 The 5-Tuple for an NFA
- 5.5 The Language Accepted by an NFA

Not A DFA



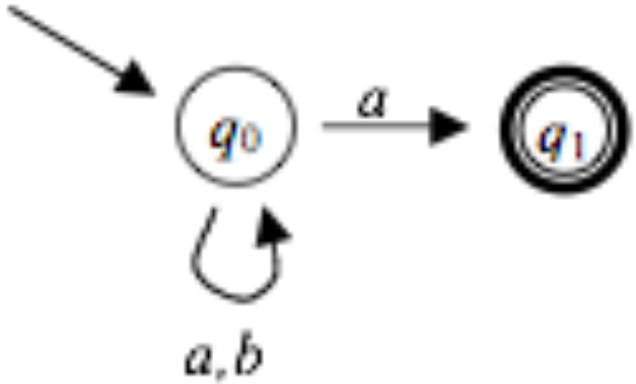
- Does not have exactly one transition from every state on every symbol:
  - Two transitions from q_0 on a
  - No transition from q_1 (on either a or b)
- Though not a DFA, this can be taken as defining a language, in a slightly different way

Possible Sequences of Moves



- We'll consider all possible sequences of moves the machine might make for a given string
- For example, on the string *aa* there are three:
  - From q_0 to q_0 to q_0, rejecting
  - From q_0 to q_0 to q_1, accepting
  - From q_0 to q_1, getting stuck on the last *a*
- Our convention for this new kind of machine: a string is in $L(M)$ if there is *at least one* accepting sequence

Nondeterministic Finite Automaton (NFA)

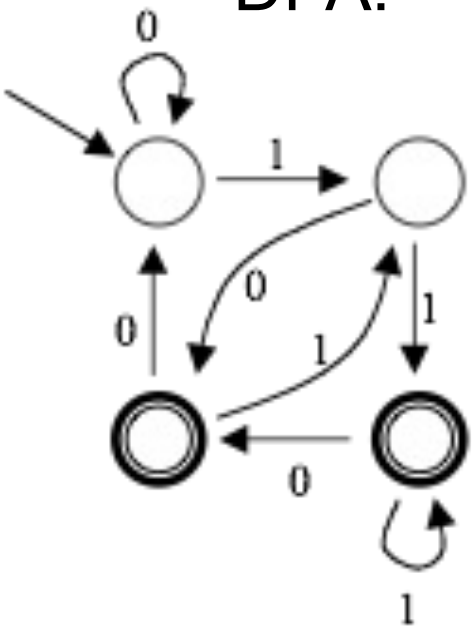


- $L(M)$ = the set of strings that have *at least one* accepting sequence
- In the example above, $L(M) = \{xa \mid x \in \{a,b\}^*\}$
- A DFA is a special case of an NFA:
  - An NFA that happens to be deterministic: there is exactly one transition from every state on every symbol
  - So there is exactly one possible sequence for every string
- NFA is *not necessarily* deterministic

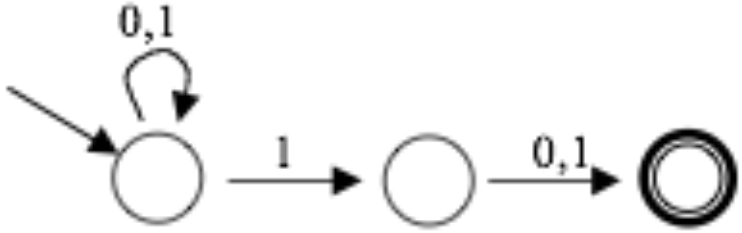
NFA Advantage

- An NFA for a language can be smaller and easier to construct than a DFA
- Strings whose next-to-last symbol is 1:

DFA:



NFA:



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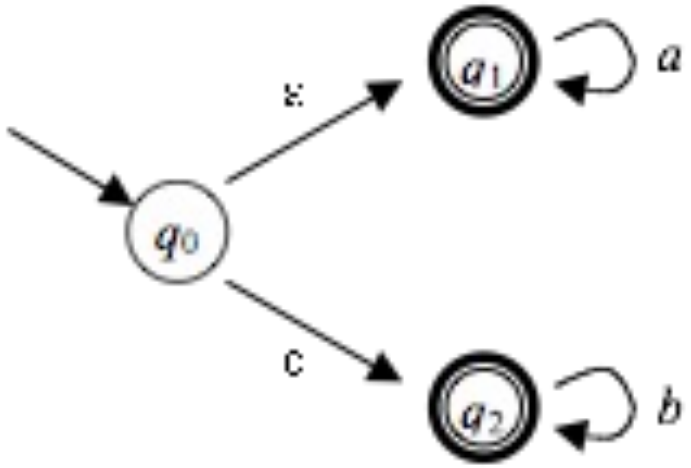
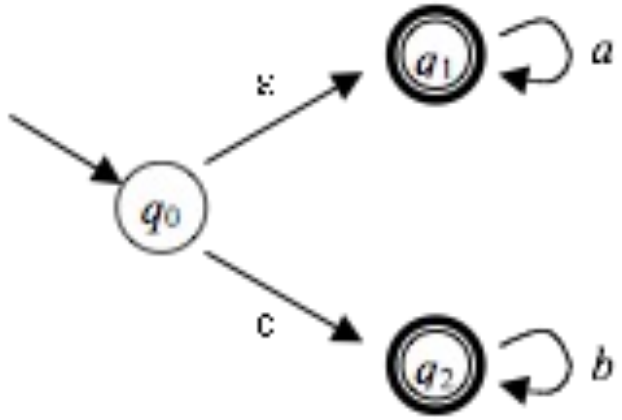
Spontaneous Transitions

- An NFA can make a state transition spontaneously, without consuming an input symbol
- Shown as an arrow labeled with ε
- For example, {*a*}* ∪ {*b*}*:

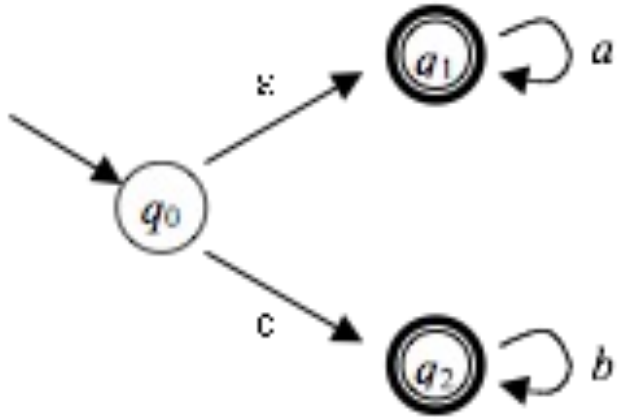


ε-Transitions To Accepting States



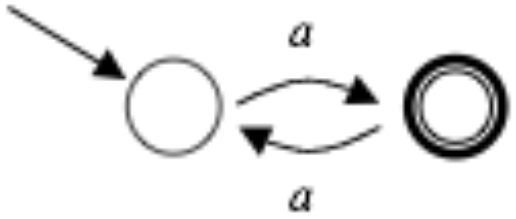
- An ε-transition can be made at any time
- For example, there are three sequences on the empty string
  - No moves, ending in q_0, rejecting
  - From q_0 to q_1, accepting
  - From q_0 to q_2, accepting
- Any state with an ε-transition to an accepting state ends up working like an accepting state too

ε-transitions For NFA Combining

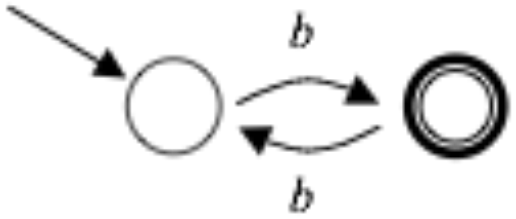


- ε-transitions are useful for combining smaller automata into larger ones
- This machine is combines a machine for {*a*}* and a machine for {*b*}*
- It uses an ε-transition at the start to achieve the union of the two languages

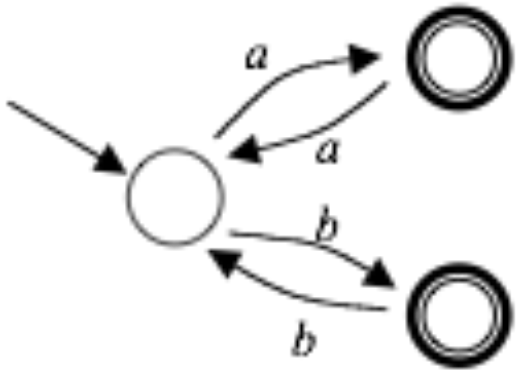
Incorrect Union



$A = \{a^n \mid n \text{ is odd}\}$

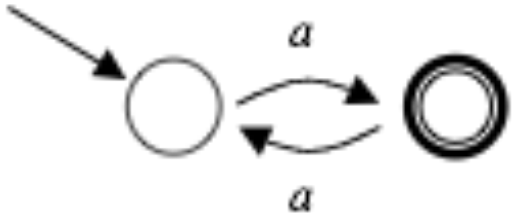


$B = \{b^n \mid n \text{ is odd}\}$

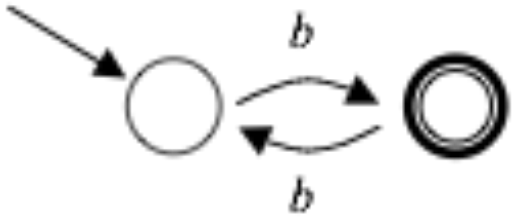


$A \cup B$?
No: this NFA accepts *aab*

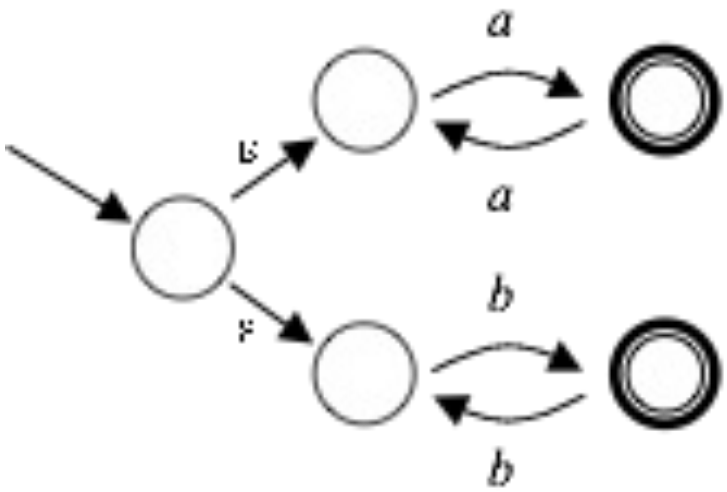
Correct Union



$A = \{a^n \mid n \text{ is odd}\}$

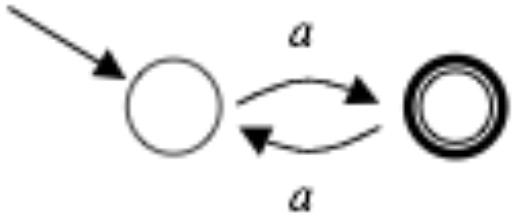


$B = \{b^n \mid n \text{ is odd}\}$

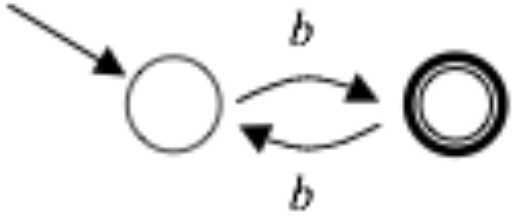


$A \cup B$

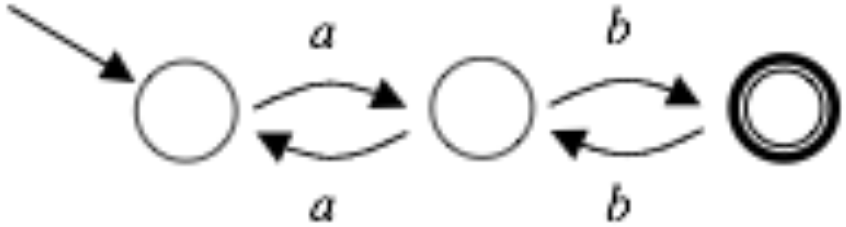
Incorrect Concatenation



$A = \{a^n \mid n \text{ is odd}\}$



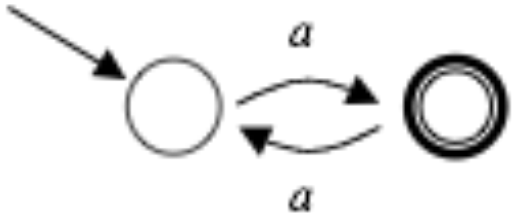
$B = \{b^n \mid n \text{ is odd}\}$



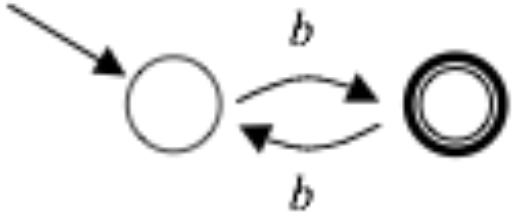
$\{xy \mid x \in A \text{ and } y \in B\}$?

No: this NFA accepts *abbaab*

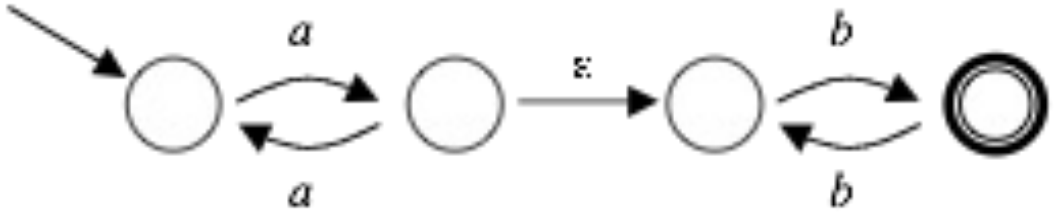
Correct Concatenation



$A = \{a^n \mid n \text{ is odd}\}$



$B = \{b^n \mid n \text{ is odd}\}$



$\{xy \mid x \in A \text{ and } y \in B\}$

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DFAs and NFAs

- DFAs and NFAs both define languages
- DFAs do it by giving a simple computational procedure for deciding language membership:
  - Start in the start state
  - Make one transition on each symbol in the string
  - See if the final state is accepting
- NFAs do it without such a clear-cut procedure:
  - Search all legal sequences of transitions on the input string?
  - How? In what order?

Nondeterminism

- The essence of nondeterminism:
  - For a given input there can be more than one legal sequence of steps
  - The input is in the language if at least one of the legal sequences says so
- We can achieve the same result by deterministically searching the legal sequences, but…
- ...this nondeterminism does not directly correspond to anything in physical computer systems
- In spite of that, NFAs have many practical applications

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Powerset

- If *S* is a set, the *powerset* of *S* is the set of all subsets of S:

$$P(S) = \{R \mid R \subseteq S\}$$

- This always includes the empty set and *S* itself
- For example,

P({1,2,3}) = {{}, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3}}

The 5-Tuple

An NFA M is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$, where:

Q is the finite set of states

Σ is the alphabet (that is, a finite set of symbols)

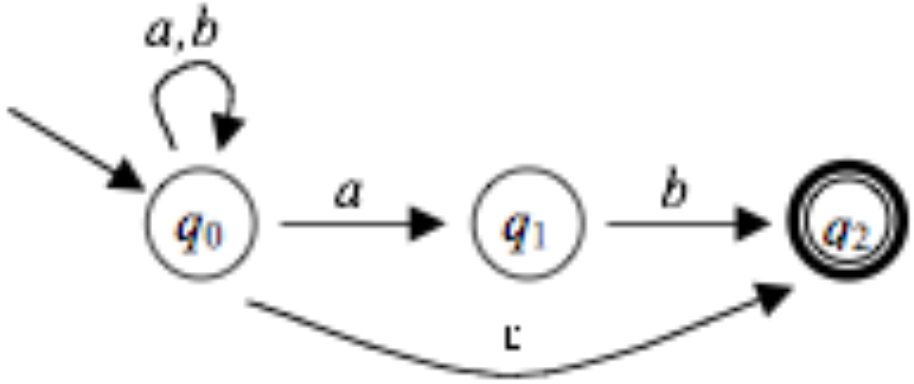
$\delta \in (Q \cdot (\Sigma \cup \{\varepsilon\}) \rightarrow P(Q))$ is the transition function

$q_0 \in Q$ is the start state

$F \subseteq Q$ is the set of accepting states

- The only change from a DFA is the transition function δ
- δ takes two inputs:
  - A state from Q (the current state)
  - A symbol from $\Sigma \cup \{\varepsilon\}$ (the next input, or ε for an ε-transition)
- δ produces one output:
  - A subset of Q (the set of possible next states)

Example:



- Formally, $M = (Q, \Sigma, \delta, q_0, F)$, where
  - $Q = \{q_0, q_1, q_2\}$
  - $\Sigma = \{a, b\}$ (we assume: it must contain at least a and b)
  - $F = \{q_2\}$
  - $\delta(q_0, a) = \{q_0, q_1\}$, $\delta(q_0, b) = \{q_0\}$, $\delta(q_0, \varepsilon) = \{q_2\}$,
    $\delta(q_1, a) = \{\}$, $\delta(q_1, b) = \{q_2\}$, $\delta(q_1, \varepsilon) = \{\}$
    $\delta(q_2, a) = \{\}$, $\delta(q_2, b) = \{\}$, $\delta(q_2, \varepsilon) = \{\}$
- The language defined is $\{a, b\}^*$

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The δ^* Function

- The δ function gives 1-symbol moves
- We'll define δ^* so it gives whole-string results (by applying zero or more δ moves)
- For DFAs, we used this recursive definition
  - $\delta^*(q,\varepsilon) = q$
  - $\delta^*(q,xa) = \delta(\delta^*(q,x),a)$
- The intuition is the similar for NFAs, but the ε-transitions add some technical hair

NFA IDs

- An *instantaneous description* (ID) is a description of a point in an NFA's execution
- It is a pair (q,x) where
  - $q \in Q$ is the current state
  - $x \in \Sigma^*$ is the unread part of the input
- Initially, an NFA processing a string x has the ID (q_0,x)
- An accepting sequence of moves ends in an ID (f,ε) for some accepting state $f \in F$

The One-Move Relation On IDs

- We write

  $$I \mapsto J$$

  if *I* is an ID and *J* is an ID that could follow from *I* after one move of the NFA
- That is, for any string $x \in \Sigma^*$ and any $\omega \in \Sigma$ or $\omega = \varepsilon$,

$$(q, \omega x) \mapsto (r, x) \text{ if and only if } r \in \delta(q, w)$$

The Zero-Or-More-Move Relation

- We write

  $I \vdash^* J$

  if there is a sequence of zero or more moves that starts with *I* and ends with *J:*

  $I \vdash \cdots \vdash J$

- Because it allows zero moves, it is a reflexive relation: for all IDs *I*,

  $I \vdash^* I$

The δ^* Function

- Now we can define the δ^* function for NFAs:

$$\delta^*(q,x) = \left\{ r \mid (q,x) \vdash^* (r,\varepsilon) \right\}$$

- Intuitively, $\delta^*(q,x)$ is the set of all states the NFA might be in after starting in state q and reading x
- Our definition allows ε-transitions, including those made before the first symbol of x is read, and those made after the last

M Accepts *x*

- Now $\delta^*(q,x)$ is the set of states *M* may end in, starting from state *q* and reading all of string *x*
- So $\delta^*(q_0,x)$ tells us whether *M* accepts *x*:

A string $x \in \Sigma^*$ is accepted by an NFA $M = (Q, \Sigma, \delta, q_0, F)$ if and only if $\delta^*(q_0, x)$ contains at least one element of *F*.

The Language An NFA Defines

For any NFA $M = (Q, \Sigma, \delta, q_0, F)$, $L(M)$ denotes the language accepted by M, which is

$$L(M) = \{x \in \Sigma^* \mid \delta^*(q_0, x) \cap F \neq \{\}\}.$$