Chapter Eleven: Non-Regular Languages

We have now encountered regular languages in several different places. They are the languages that can be recognized by a DFA. They are the languages that can be recognized by an NFA. They are the languages that can be denoted by a regular expression. They are the languages that can be generated by a right-linear grammar. You might begin to wonder: are there any languages that are not regular?

In this chapter, we will see that there are. There is a proof tool that is often used to prove languages non-regular. It is called the pumping lemma, and it describes an important property that all regular languages have. If you can show that a given language does not have this property, you can conclude that it is not a regular language.

Outline

- 11.1 The Language $\{a^n b^n\}$
- 11.2 The Languages $\{xx^R\}$
- 11.3 Pumping
- 11.4 Pumping-Lemma Proofs
- 11.5 Strategies
- 11.6 Pumping And Finite Languages

The Language $\{a^n b^n\}$

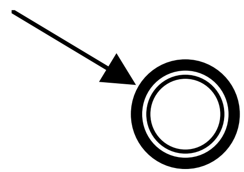
- Any number of *a*s followed by the same number of *b*s
- Easy to give a grammar for this language:

$$S \rightarrow aSb \mid \varepsilon$$

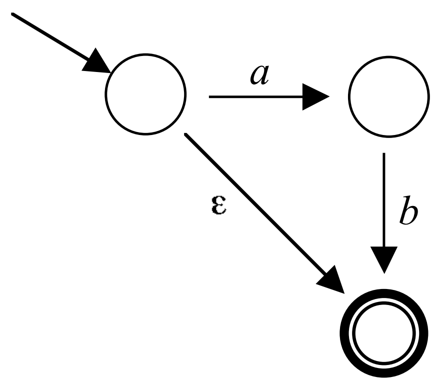
- All derivations of a fully terminal string use the first production n=0 or more times, then the last production once: $a^n b^n$
- Is it a regular language? For example, is there an NFA for it?

Trying To Build An NFA

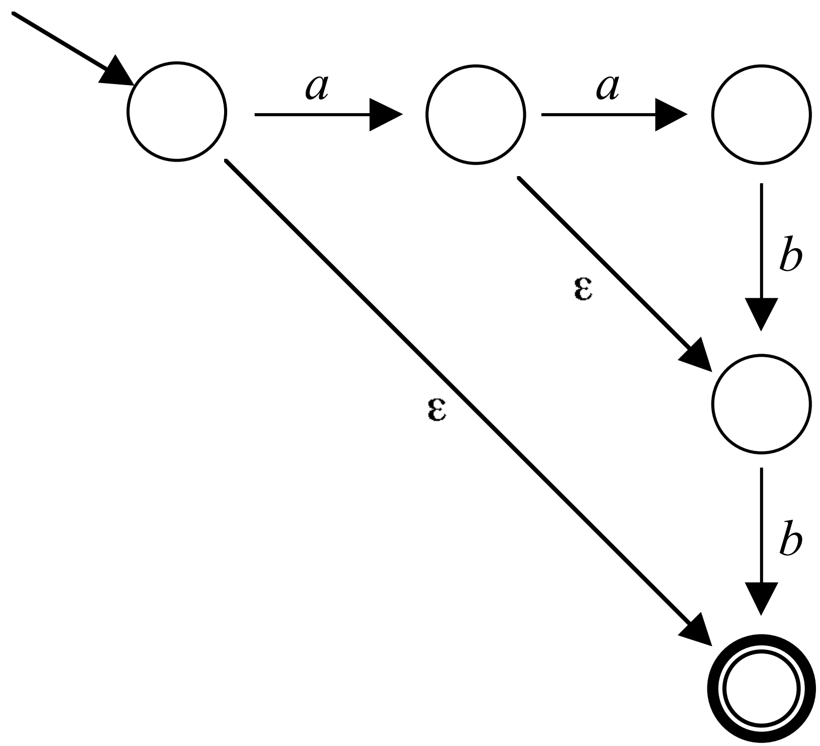
- We'll try working up to it
- The subset $\{a^n b^n \mid n \leq 0\}$:



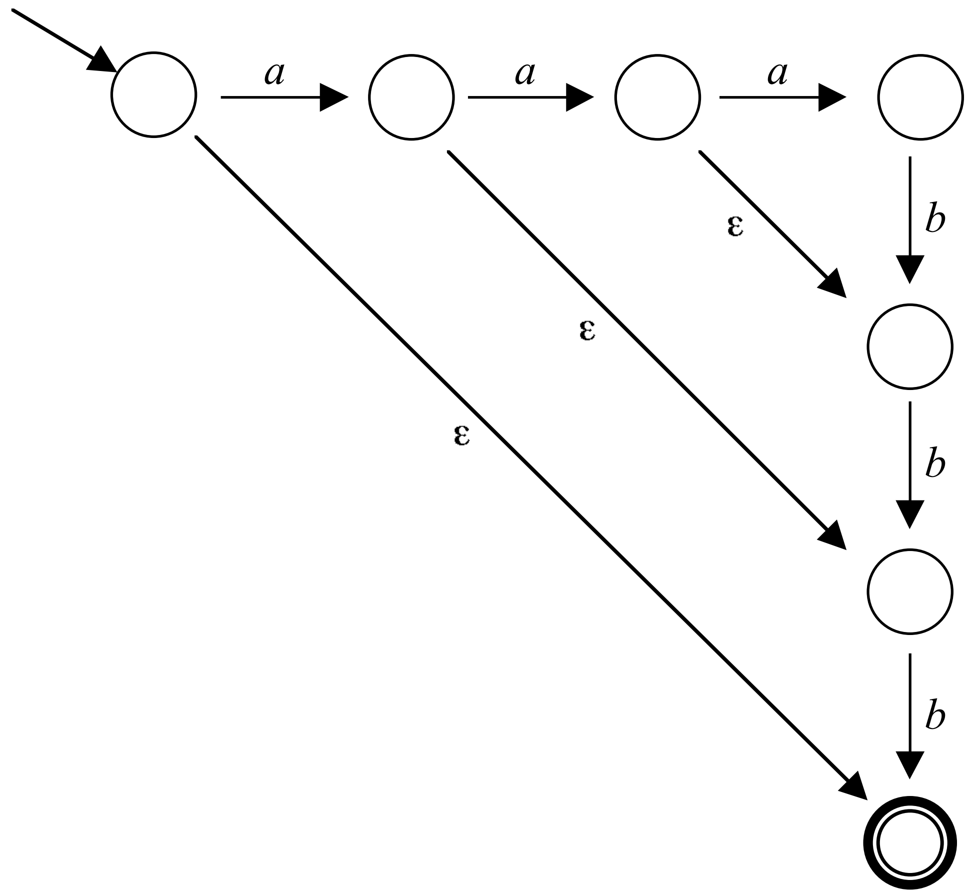
- The subset $\{a^n b^n \mid n \leq 1\}$:



The Subset $\{a^n b^n \mid n \leq 2\}$



The Subset $\{a^n b^n \mid n \le 3\}$



A Futile Effort

- For each larger value of *n* we added two more states
- We're using the states to count the *a*s, then to check that the same number of *b*s follow
- That's not going to be a successful pattern on which to build an NFA for all of $\{a^n b^n\}$
  - NFA needs a fixed, finite number of states
  - No fixed, finite number will be enough to count the unbounded *n* in $\{a^n b^n\}$
- This is *not* a proof that no NFA can be constructed
- But it does contain the germ of an idea for a proof…

Theorem 11.1

The language $\{a^n b^n\}$ is not regular.

- Let $M = (Q, \{a,b\}, \delta, q_0, F)$ be any DFA over the alphabet $\{a,b\}$; we'll show that $L(M) \neq \{a^n b^n\}$
- Given *a*s for input, *M* visits a sequence of states:
  - $\delta^*(q_0,\varepsilon)$, then $\delta^*(q_0,a)$, then $\delta^*(q_0,aa)$, and so on
- Since *Q* is finite, *M* eventually revisits one:
  - $\exists$ i and j with $i < j$ such that $\delta^*(q_0,a^i) = \delta^*(q_0,a^j)$
- Append b^i, and we see that $\delta^*(q_0,a^i b^i) = \delta^*(q_0,a^j b^i)$
- So *M* either accepts both $a^i b^i$ and $a^j b^i$, or rejects both
- $\{a^n b^n\}$ contains $a^i b^i$ but not $a^j b^i$, so $L(M) \neq \{a^n b^n\}$
- So no DFA has $L(M) = \{a^n b^n\}$: $\{a^n b^n\}$ is not regular

A Word About That Proof

- Nothing was assumed about the DFA *M*, except its alphabet {*a*,*b*}
- In spite of that, we were able to infer quite a lot about its behavior
- The basic insight: with a sufficiently long string we can force any DFA to repeat a state
- That's the basis of a wide variety of non-regularity proofs

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The Languages $\{xx^R\}$

- The notation x^R means the string x, reversed
- $\{xx^R\}$ is the set of strings that can be formed by taking any string in Σ^*, and appending the same string, reversed
- For $\Sigma = \{a,b\}$, $\{xx^R\}$ includes the strings ε, *aa, bb, abba, baab, aaaa, bbbb*, and so on
- Another way of saying it: $\{xx^R\}$ is the set of even-length palindromes

A Grammar For $\{xx^R \mid x \in \{a,b\}^*\}$

$$S \rightarrow aSa \mid bSb \mid \varepsilon$$

- A derivation for *abba*:
  - $S \Rightarrow aSa \Rightarrow abSba \Rightarrow abba$
- A derivation for *abaaba*:
  - $S \Rightarrow aSa \Rightarrow abSba \Rightarrow abaSaba \Rightarrow abaaba$
- Every time you use one of the first two productions, you add a symbol to the end of the first half, and the same symbol to the start of the second half
- So the second half is always the reverse of the first half: $L(G) = \{xx^R \mid x \in \{a,b\}^*\}$
- But is this language regular?

Intuition

- After seeing the first example, you may already have the feeling this can't be regular
  - A finite state machine would have to use states to keep track of x, then check that it is followed by a matching x^R
  - But there is no bound on the length of x, so no fixed, finite number of states will suffice
- The formal proof is very similar to the one we used for $\{a^n b^n\}$…

Theorem 11.2

The language $\{xx^R\}$ is not regular for any alphabet with at least two symbols.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be any DFA with $|\Sigma| \geq 2$; we'll show that $L(M) \neq \{xx^R\}$
- Σ has at least two symbols; call two of these *a* and *b*
- Given *a*s for input, *M* visits a sequence of states:
  - $\delta^*(q_0,\varepsilon)$, then $\delta^*(q_0,a)$, then $\delta^*(q_0,aa)$, and so on
- Since *Q* is finite, *M* eventually revisits one:
  - $\exists$ *i* and *j* with $i < j$ such that $\delta^*(q_0,a^i) = \delta^*(q_0,a^j)$
- Append bba^j, and we see that $\delta^*(q_0,a^ibba^j) = \delta^*(q_0,a^jbba^j)$
- So *M* either accepts both a^ibba^j and a^jbba^j, or rejects both
- $\{xx^R\}$ contains a^jbba^j but not a^ibba^j, so $L(M) \neq \{xx^R\}$
- So no DFA has $L(M) = \{xx^R\}$: $\{xx^R\}$ is not regular

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Review

- We've shown two languages non-regular: $\{a^nb^n\}$ and $\{xx^R\}$
- In both cases, the key idea was to choose a string long enough to make any given DFA repeat a state
- For both those proofs we just used strings of *a*s, and showed that ∃ i and j with $i < j$ such that $\delta^*(q_0,a^i) = \delta^*(q_0,a^j)$

Multiple Repetitions

- When you've found a state that repeats once, you can make it repeat again and again
- For example, our $\delta^*(q_0,a^i) = \delta^*(q_0,a^j)$:
  - Let r be the state in question: $r = \delta^*(q_0,a^i)$
  - After j-i more as it repeats: $r = \delta^*(q_0,a^{i+(j-i)})$
  - That little substring $a^{(j-i)}$ takes it from state r back to state r
  - $r = \delta^*(q_0,a^i)$
    $= \delta^*(q_0,a^{i+(j-i)})$
    $= \delta^*(q_0,a^{i+2(j-i)})$
    $= \delta^*(q_0,a^{i+3(j-i)})$

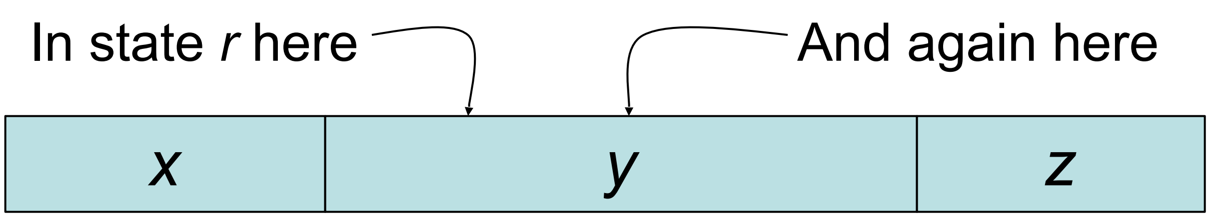
Pumping

- We say that the substring $a^{(j-i)}$ can be *pumped* any number of times, and the DFA always ends up in the same state
- All regular languages have an important property involving pumping
- Any sufficiently long string in a regular language must contain a pumpable substring
- Formally, the pumping lemma…

Lemma 11.3: The Pumping Lemma for Regular Languages

For all regular languages L there exists some integer k such that for all $xyz \in L$ with $|y| \geq k$, there exist $uvw = y$ with $|v| > 0$, such that for all $i \geq 0$, $xuv^iwz \in L$.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be any DFA with $L(M) = L$
- Choose $k = |Q|$
- Consider any x, y, and z with $xyz \in L$ and $|y| \geq k$
- Let r be a state that repeats during the y part of xyz
  - We know such a state exists because we have $|y| \geq |Q|$…



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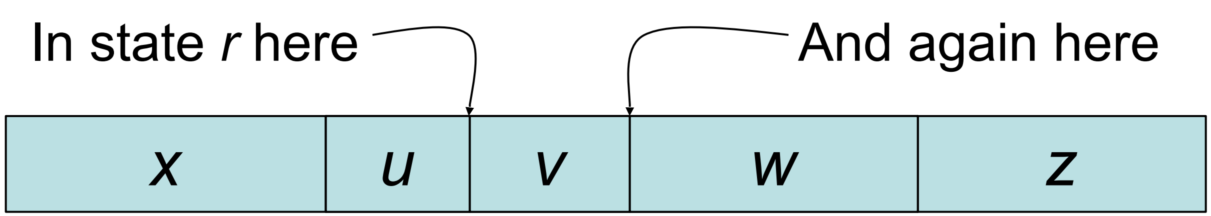
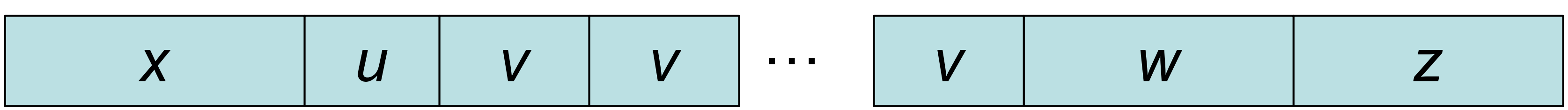
- Let $M = (Q, \Sigma, \delta, q_0, F)$ be any DFA with $L(M) = L$
- Choose $k = |Q|$
- Consider any x, y, and z with $xyz \in L$ and $|y| \geq k$
- Let r be a state that repeats during the y part of xyz
- Choose $uvw = y$ so that $\delta^*(q_0,xu) = \delta^*(q_0,xuv) = r$
- Now v is pumpable: for all $i \geq 0$, $\delta^*(q_0,xuv^i) = r$…



Lemma 11.3: The Pumping Lemma for Regular Languages

For all regular languages L there exists some integer k such that for all $xyz \in L$ with $|y| \geq k$, there exist $uvw = y$ with $|v| > 0$, such that for all $i \geq 0$, $xuv^iwz \in L$.

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be any DFA with $L(M) = L$
- Choose $k = |Q|$
- Consider any x, y, and z with $xyz \in L$ and $|y| \geq k$
- Let r be a state that repeats during the y part of xyz
- Choose $uvw = y$ so that $\delta^*(q_0, xu) = \delta^*(q_0, xuv) = r$
- Now v is pumpable: for all $i \geq 0$, $\delta^*(q_0, xuv^i) = r$
- Then for all $i \geq 0$, $\delta^*(q_0, xuv^iwz) = \delta^*(q_0, xuvwz) = \delta^*(q_0, xyz) \in F$
- Therefore, for all $i \geq 0$, $xuv^iwz \in L$



Pumping Lemma Structure

For all regular languages L there exists some integer k such that for all $xyz \in L$ with $|y| \geq k$, there exist $uvw = y$ with $|v| > 0$, such that for all $i \geq 0$, $xuv^iwz \in L$.

- Notice the alternating "for all" and "there exist" clauses:
  1. $\forall L$ …
  2. $\exists k$ …
  3. $\forall xyz$ …
  4. $\exists uvw$ …
  5. $\forall i$ …
- Our proof showed how to construct the ∃ parts
- But that isn't part of the lemma: it's a black box
- The lemma says only that *k* and *uvw* exist

Outline

Pumping-Lemma Proofs

- The pumping lemma is very useful for proving that languages are not regular
- For example, $\{a^n b^n\}$…

$\{a^n b^n\}$ Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $L = \{a^n b^n\}$ is regular, so the pumping lemma holds for L. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k$

   $y = b^k$

   $z = \varepsilon$

   Now $xyz = a^k b^k \in L$ and $|y| \geq k$ as required.
3. Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^iwz \in L$.
4. Choose $i = 2$. Since v contains at least one b and nothing but bs, uv^2w has more bs than uvw. So xuv^2wz has more bs than as, and so $xuv^2wz \notin L$.
5. By contradiction, $L = \{a^n b^n\}$ is not regular.

The Game

- The alternating ∀ and ∃ clauses of the pumping lemma make these proofs a kind of game
- The ∃ parts (k and uvw) are the pumping lemma's moves: these values exist, but are not ours to choose
- The ∀ parts (L, xyz, and i) are our moves: the lemma holds for all proper values, so we have free choice
- We make our moves strategically, to force a contradiction
- No matter what the pumping lemma does with its moves, we want to end up with some $xuv^iwz \notin L$

The Pattern

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $L = \{a^n b^n\}$ is regular, so the pumping lemma holds for L. Let k be as given by the pumping lemma.
2.

Here, you chose xyz and show that they meet the requirements, $xyz \in L$ and $|y| \geq k$. Choose them so that pumping in the y part will lead to a contradiction, a string $\notin L$.

3 Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^iwz \in L.$

4

Here, you choose i, the number of times to pump, and show that you have a contradiction: $xuv^iwz \notin L$.

5 By contradiction, $L = \{a^n b^n\}$ is not regular.

$\{xx^R\}$ Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $L = \{xx^R\}$ is regular, so the pumping lemma holds for L. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k bb$

   $y = a^k$

   $z = \varepsilon$

   Now $xyz = a^k bba^k \in L$ and $|y| \geq k$ as required.
3. Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^i wz \in L$.
4. Choose $i = 2$. Since v contains at least one a and nothing but as, uv^2w has more as than uvw. So xuv^2wz has more as after the bs than before them, and thus $xuv^2wz \notin L$.
5. By contradiction, $L = \{xx^R\}$ is not regular.

Outline

Proof Strategy

- It all comes down to those four delicate choices: *xyz* and *i*
- Usually, there are a number of choices that successfully lead to a contradiction
- And, of course many others that fail
- For example: let $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$
- We'll try a pumping-lemma proof that *A* is not regular

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = aaa$

   $y = b$

   $z = aaa$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = aaa$

   $y = b$

   $z = aaa$

Bad choice. The pumping lemma requires $|y| \geq k$. It never applies to fixed-size examples. Since k is not known in advance, y must be some string that is constructed using k, such as a^k.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = a^k$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = a^k$

Bad choice. The pumping lemma lemma only applies if the string $xyz \in A$. That is not the case here.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^n$

   $y = b$

   $z = a^n$

   ?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

$x = a^n$

$y = b$

$z = a^n$

This is ill-formed, since the value of n is not defined. At this point the only integer variable that is defined is k.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k$

   $y = b^{k+2}$

   $z = a^k$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

$x = a^k$

$y = b^{k+2}$

$z = a^k$

This meets the requirements $xyz \in A$ and $|y| \geq k$, but it is a bad choice because it won't lead to a contradiction. Pumping within the string y will change the number of bs in the middle, but the resulting string can still be in A.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k$

   $y = bba^k$

   $z = \varepsilon$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

$x = a^k$

$y = bba^k$

$z = \varepsilon$

This meets the requirements $xyz \in A$ and $y| \geq k$, but it is a bad choice because it won't lead to a contradiction. The pumping lemma can choose any $uvw = y$ with $|v| > 0$. If it chooses $u=b$, $v=b$, and $w = a^k$, there will be no contradiction, since for all $i \geq 0$,
$xuv^iwz \in A$.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k b$

   $y = a^k$

   $z = \varepsilon$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = a^k b$

   $y = a^k$

   $z = \varepsilon$

Good choice. It meets the requirements $xyz \in A$ and $|y| \geq k$, and it will lead to a contradiction because pumping anywhere in the y part will change the number of *a*s after the *b*, without changing the number before the *b*.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = ba^k$

?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = ba^k$

An equally good choice.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = ba^k$

   Now $xyz = a^k b a^k \in A$ and $|y| \geq k$ as required.

3 Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^iwz \in A$.

1. Choose $i = 1$

   ?

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = ba^k$

   Now $xyz = a^k b a^k \in A$ and $|y| \geq k$ as required.

3 Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^i wz \in A$.

1. Choose $i = 1$

Bad choice -- the only bad choice for i in this case! When $i = 1$, $xuv^i wz \in A$, so there is no contradiction.

A Is Not Regular

1. Proof is by contradiction using the pumping lemma for regular languages. Assume that $A = \{a^n b^j a^n \mid n \geq 0, j \geq 1\}$ is regular. Let k be as given by the pumping lemma.
2. Choose x, y, and z as follows:

   $x = \varepsilon$

   $y = a^k$

   $z = ba^k$

   Now $xyz = a^k b a^k \in A$ and $|y| \geq k$ as required.

3 Let u, v, and w be as given by the pumping lemma, so that $uvw = y$, $|v| > 0$, and for all $i \geq 0$, $xuv^i wz \in A$.

4 Choose $i = 2$. Since v contains at least one a and nothing but as, uv^2w has more as than uvw. So xuv^2wz has more as before the b than after it, and thus $xuv^2wz \notin A$.

5 By contradiction, A is not regular.

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What About Finite Languages?

For all regular languages L there exists some integer k such that for all $xyz \in L$ with $|y| \geq k$, there exist $uvw = y$ with $|v| > 0$, such that for all $i \geq 0$, $xuv^iwz \in L$.

- The pumping lemma applies in a trivial way to any finite language L
- Choose k greater than the length of the longest string in L
- Then it is clearly true that "for all $xyz \in L$ with $|y| \geq k$, …" since there are *no* strings in L with $|y| \geq k$
- It is *vacuously true*
- In fact, all finite languages are regular…

Theorem 11.6

All finite languages are regular.

- Let A be any finite language of n strings: $A = \{x_1, ..., x_n\}$
- There is a regular expression that denotes this language: $A = L(x_1 + ... + x_n)$
- Or, in case $n = 0$, $A = L(\emptyset)$
- Since A is denoted by a regular expression, A is a regular language