MATH 103 Pre-Calculus Mathematics
Quiz #8 Fall 2008
Sample Solutions

1. Describe a polynomial Q having integer coefficients (preferably as small as possible) and having the same zeros as the polynomial

$$P(x) = \frac{2}{9}x^4 - \frac{1}{6}x^3 + x^2 + \frac{3}{4}x - 1$$

Solution: Keeping in mind that, for any function f and any non-zero constant a, the function f has the same zeros as the function $g(x) = a \cdot f(x)$, all we need is to find a constant by which to multiply P such that all the resulting coefficients are integers. The smallest such constant is the least common multiple of the denominators of P's coefficients.

The denominators of the coefficients of P are 9, 6, 1, 4 and 1. Their least common multiple is 36. Hence we multiply P by 36 to get

$$Q(x) \;=\; 36 \cdot P(x) \;=\; 8x^4 - 6x^3 + 36x^2 + 27x - 36$$

2. List all rational numbers (but try to avoid duplicates) that, according to the *Rational Zero Test*, are candidates for being zeros of the polynomial

$$P(x) = 2x^3 - 4x^2 - 2x + 4$$

Solution: The Rational Zero Test says that p/q (assumed to be in simplest form) cannot be a zero of a polynomial with integer coefficients unless p is a divisor of its constant term and q is a divisor of its leading coefficient.

Here we have a constant term of 4 and a leading coefficient of 2. Hence, p must be one of ± 1, ± 2, or ± 4 and q must be either ± 1 or ± 2. The candidates are thus those in the set

$$\{\pm 1, \pm\frac{1}{2}, \pm 2, \pm 4\}$$

3. Factor the polynomial $P(x) = x^3 - 2x^2 - 2x + 4$ completely. Recall that the quadratic formula tells us that any zeros of $ax^2 + bx + c$ satisfy

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solution: P is a polynomial with integer coefficients, and so we can apply the Rational Zero Test to obtain a set including every possible rational zero of P. Doing so, we get the set $\{\pm 1, \pm 2, \pm 4\}$.

By evaluating P at 2 (probably after trying 1 and -1 and finding that neither is a zero of P), we find that it is a zero of P. It follows, by the Factor Theorem, that $x - 2$ is a factor of P.

Now (possibly after trying -2 and ± 4 and finding none of them to be zeros) we divide $x - 2$ into P (using the polynomial division algorithm described in the textbook) and obtain as a quotient $x^2 - 2$. (The remainder is zero, of course, or else $x - 2$ would not be a factor of P!)

At this point, we recall that, for any positive constant a, $x^2 - a = (x - \sqrt{a})(x + \sqrt{a})$. So we get

$$P(x) = x^3 - 2x^2 - 2x + 4 = (x - 2)(x^2 - 2) = (x - 2)(x - \sqrt{2})(x + \sqrt{2})$$

and we are done. Or, had we not remembered that bit of factoring wisdom, we may have reasoned as follows: A zero of $x^2 - 2$ corresponds to a solution of the equation $x^2 - 2 = 0$. Adding two to both sides, we get $x^2 = 2$, which has as solutions $x = \pm\sqrt{2}$, which means (by the Factor Theorem) that $x - \sqrt{2}$ and $x + \sqrt{2}$ are factors of $x^2 - 2$. So we arrive at the same solution as was shown above.

If neither of the insights described above occur to us, we can turn to the Rational Zero Test, which tells us that the only possible rational zeros of $x^2 - 2$ are ± 1 and ± 2. Trying them, we find that none are zeros. (Note that any value previously found not to be a zero of P need not be considered as a possible zero of $x^2 - 2$, because any zero of the latter is also a zero of the former.) From that we draw the conclusion that any as-yet-unidentified zeros of P must be irrational.

So we turn to the quadratic formula, which is applicable because $x^2 - 2$ is, indeed, quadratic (i.e., of degree two). Applying that, we find that $x^2 - 2 = 0$ when either $x = \sqrt{2}$ or $x = -\sqrt{2}$. (As part of this calculation, we have to recognize that $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$.) It follows from the Factor Theorem that both $x - \sqrt{2}$ and $x + \sqrt{2}$ are factors of $x^2 - 2$, and hence of P, and so we get the same result as above.