MATH 103 Pre-Calculus Mathematics
Test #2 Fall 2008
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Sample Solutions

1. The table below defines the functions f and g, each of which has as its domain the set of integers $\{0,1,2,3,4\}$. Fill in the missing entries in the table, using **UND** to mean "undefined". (For example, in the first row you are to supply the values $(f+g)(0)$, $(f-g)(0)$, etc.)

Also answer the questions underneath the table.

Solution: The entries in the first three columns were given; the remaining entries (in boldface) are the answers.

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f-g)(x)$	$(f \cdot g)(x)$	$(f/g)(x)$	$(f \circ g)(x)$	$(g \circ f)(x)$
0	1	4	**5**	**-3**	**4**	**1/4**	**2**	**0**
1	-1	0	**-1**	**-1**	**0**	**UND**	**1**	**UND**
2	0	1	**1**	**-1**	**0**	**0**	**-1**	**4**
3	3	8	**11**	**-5**	**24**	**3/8**	**UND**	**8**
4	2	1	**3**	**1**	**2**	**2**	**-1**	**1**

What is the domain of $(g \circ f)$?
Answer: $\{0,2,3,4\}$.
Explanation: Recall that, for x_1 to be in the domain of $(g \circ f)$ requires that it be in the domain of f and that its "image" under f, $f(x_1)$, be in the domain of g. The first condition limits the domain of $(g \circ f)$ to be a subset of the domain of f, which is (as stated in the problem) $\{0,1,2,3,4\}$. But for each element x_1 of this set, whether or not $f(x_1)$ is in the domain of g (which is equivalent to $g(f(x_1))$ being defined) can be determined by looking in the last column of the table! Doing so, we see that $(g \circ f)$ is defined for all values in the set $\{0,1,2,3,4\}$, except 1. Hence, its domain is $\{0,1,2,3,4\} - \{1\}$, or $\{0,2,3,4\}$.

What is its range?
Answer: $\{0,1,4,8\}$
Explanation: We have already established that the domain of $(g \circ f)$ is $\{0,2,3,4\}$. Hence, its range is $\{(g \circ f)(0), (g \circ f)(2), (g \circ f)(3), (g \circ f)(4)\}$ (i.e., the set of values produced by applying $(g \circ f)$ to all the elements of its domain). But this is just the set of values appearing in the last column of the table!

2. Suppose that the graph of $y = g(x)$ is obtained from that of $y = f(x)$ by vertically elongating the latter by a factor of 3 and then shifting it four units to the left and seven units down. Describe function g in terms of function f.

Solution: $g(x) = 3 \cdot f(x+4) \;-\; 7$

3. Below are four graphs of $y = f(x)$, labeled Figure 1 through Figure 4.
In Figure 1, sketch the graph of $y = g(x) = |f(x)|$.
In Figure 2, sketch the graph of $y = g(x) = -\frac{1}{2}f(x)$.
In Figure 3, sketch the graph of $y = g(x) = \frac{1}{f(x)}$.
In Figure 4, relabel the tick marks on the x-axis to obtain a graph of $y = g(x) = f(2x)$.

Solution: The figures are on the following page. Here we explain how they were obtained.

Figure 1: To obtain the graph of g from that of f, we begin with the graph of f and we take any section of it that lies below the x-axis and reflect it about the x-axis. This corresponds to the fact that, if the point (x_1, y_1) lies on the graph of f (which is to say that $f(x_1) = y_1$), then $(x_1, |y_1|)$ lies on the graph of g. In the case $y_1 \geq 0$, (x_1, y_1) and $(x_1, |y_1|)$ are the same point. But in the case $y_1 < 0$, $(x_1, |y_1|)$ is $(x_1, -y_1)$.

Figure 2: To obtain the graph of g from that of f, we vertically compress the latter (so that every point (x_1, y_1) is replaced by $(x_1, y_1/2)$) and then reflect it about the x-axis so that the latter point is replaced by $(x_1, -y_1/2)$.

Figure 3: See the box on page 96 (and the accompanying Figure 9) describing how to graph the reciprocal of a function.

Figure 4: We observe that, for every point (x_1, y_1) on the graph of f, there is a "corresponding" point $x_1/2, y_1)$ on the graph of g. Why? Because $g(x_1/2) = f(2 \cdot x_1/2) = f(x_1) = y_1$. Hence, to obtain the graph of g from that of f, we horizontally compress the latter by a factor of 2. This corresponds to how the tick marks on the x-axis are labeled in Figure 4 below (so that the distance between adjacent marks is one-half rather than one).

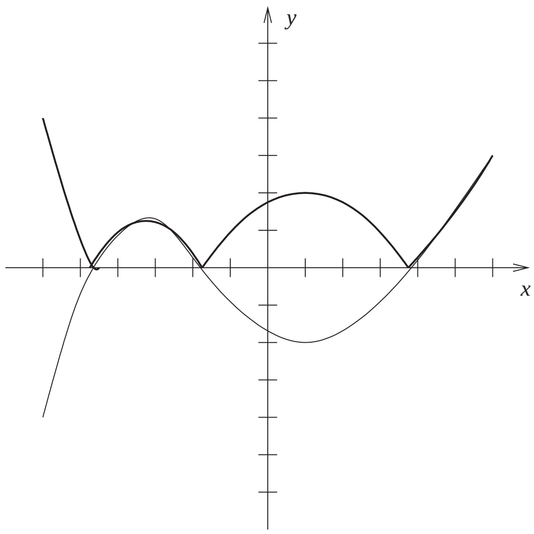

Figure 1: $y = f(x)$ and $y = g(x) = |f(x)|$

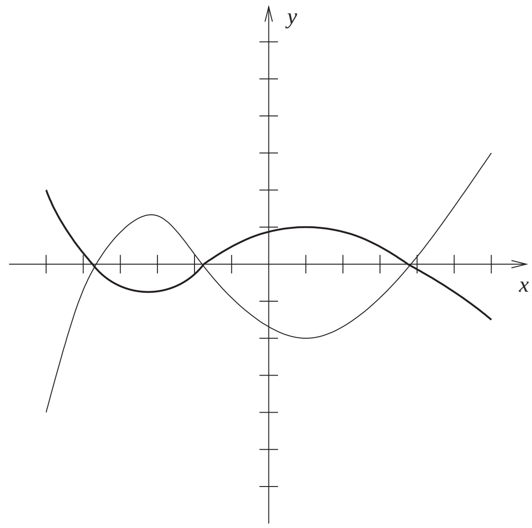

Figure 2: $y = f(x)$ and $y = g(x) = -\frac{1}{2}f(x)$

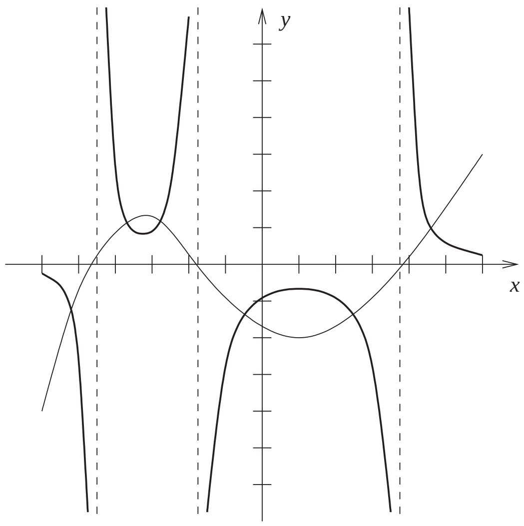

Figure 3: $y = f(x)$ and $y = g(x) = \frac{1}{f(x)}$

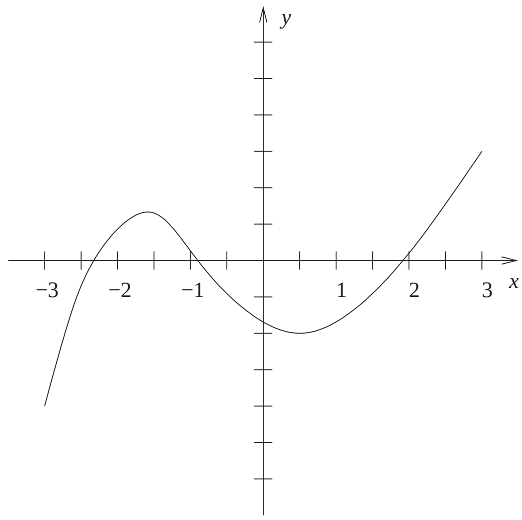

Figure 4: $y = g(x) = f(2x)$

4. For each function f, express it as a composition of two or more of the functions $g(x) = 1/x$, $u(x) = 2x$, $v(x) = x + 3$, $w(x) = \sqrt{x}$, and $h_n(x) = x^n$ (where n is any positive integer). For example, if $f(x) = 2 \cdot (x + 3)^5$, then $f = (u \circ h_5 \circ v)$. Verify at least one of your answers and, preferably, more of them.

(a) $f(x) = x^2 + 3$

(b) $f(x) = (x + 3)^2$

(c) $f(x) = (2x + 3)^4$

(d) $f(x) = \sqrt{2/(x + 3)}$

(e) $f(x) = 4x$

Solution:

(a) $f = (v \circ h_2)$

(b) $f = (h_2 \circ v)$

(c) $f = (h_4 \circ v \circ u)$

(d) $f = (w \circ u \circ g \circ v)$

(e) $f = (u \circ u)$

Verifications:

(a)

$$(v \circ h_2)(x) = v(h_2(x)) = v(x^2) = x^2 + 3$$

(b)

$$(h_2 \circ v)(x) = h_2(v(x)) = h_2(x + 3) = (x + 3)^2$$

(c)

$$(h_4 \circ v \circ u)(x) = h_4(v(u(x))) = h_4(v(2x)) = h_4(2x + 3) = (2x + 3)^4$$

(d)

$$(w \circ u \circ g \circ v)(x) = w(u(g(v(x)))) = w(u(g(x+3))) = w(u\left(\frac{1}{x+3}\right)) = w\left(\frac{2}{x+3}\right) = \sqrt{2/(x + 3)}$$

(e)

$$(u \circ u)(x) = u(u(x)) = u(2x) = 4x$$

5. For each function, either give evidence that it is not one-to-one or else find its inverse.

(a) $f(x) = -2x + 1$

(b) $f(x) = |x - 1| + 5$

(c) $f(x) = 2\sqrt[3]{x+4} - 1$

(d) $f(x) = x^2 + 2x - 2$

Solution

(a) This function clearly describes a line with slope -2. Hence, it is one-to-one and, therefore, has an inverse. To apply f to a value you would multiply it by -2 and then add one. To invert that, you would subtract one and then divide by -2, which is described by the function $g(x) = (x - 1)/2$.

To verify that g is, indeed, the inverse of f, we show that $(g \circ f)(x) = x$ for all x:

$$(g \circ f)(x) = g(f(x)) = g(-2x + 1) = \frac{(-2x + 1) - 1}{-2} = \frac{-2x}{-2} = x$$

(b) The graph of this function is like that of $y = |x|$, except shifted one unit to the right and five units up. Recalling what the graph of $y = |x|$ looks like (lines of slopes -1 and 1 meeting at the origin), it is clear that f fails the horizontal line test and hence is not one-to-one. To be a bit more rigorous, we could point out that $f(0) = 6 = f(2)$ in order to demonstrate that f is not one-to-one. (Indeed, because the graph of f has symmetry with respect to the vertical line $x = 1$ (you should be able to figure out why), we have $f(1 - c) = f(1 + c)$ for every constant c.)

(c) To apply f, you would add 4, take the cube root, multiply by 2, and subtract 1. To invert that, you would add 1, divide by 2, cube (i.e., raise to the third power), and then subtract 4. This sequence of steps is described by the function $g(x) = \left(\frac{x+1}{2}\right)^3 - 4$.

Now we verify that g is the inverse of f:

$$(g \circ f)(x) = g(f(x)) = g(2\sqrt[3]{x+4} - 1) = \left(\frac{(2\sqrt[3]{x+4} - 1) + 1}{2}\right)^3 - 4 = \left(\sqrt[3]{x+4}\right)^3 - 4 = (x+4) - 4 = x$$

(d) As f is described by a quadratic expression, we observe that its graph is that of a parabola and hence it does not pass the horizontal line test. It follows that f is not one-to-one.

To be a little more rigorous, we could use the *complete the square* method to get that $f(x) = (x + 1)^2 - 3$, which means that the graph of f is an upward-opening parabola with vertex $(-1, -3)$. This means that the graph is symmetric with respect to the vertical line $x = -1$ and so $f(-1 - c) = f(-1 + c)$ for every constant c. Taking $c = 1$, we get $f(-2) = -2 = f(0)$ to demonstrate that f is not one-to-one.

6. Let $f(x) = \frac{1}{(x-1)(x+1)}$ and $g(x) = \sqrt{x}$.

(a) What is the domain of f?

(b) What is the domain of g?

(c) What is the domain of $(g \circ f)$?

(d) What is the domain of $(f \circ g)$?

Solution:

(a) The domain of f includes every real number that, when plugged in for x, results in a non-zero denominator. That is the set $\mathcal{R} - \{-1, 1\}$, or, expressed using intervals, $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

(b) The domain of g is the set of nonnegative real numbers, or $[0, \infty)$. (Negative numbers are excluded because the square root of such a number is undefined.)

(c) The domain of $(g \circ f)$ includes any number x that is in the domain of f and is such that $f(x)$ is in the domain of g. The first condition excludes -1 and 1. The second condition excludes any x such that $f(x)$ is negative. From an analysis of f (similar to many that we've done in this course) we find that $f(x) < 0$ iff x is in the interval $(-1, 1)$. So the domain of $(g \circ f)$ is

$$\mathcal{R} - \{-1, 1\} - (-1, 1) = \mathcal{R} - [-1, 1] = (-\infty, 1) \cup (1, \infty)$$

(d) The domain of $(f \circ g)$ includes any number x that is in the domain of g and is such that $g(x)$ is in the domain of f. The first condition excludes all negative reals. The second condition excludes any x such that either $g(x) = -1$ (which is impossible) or $g(x) = 1$ (which is the case only when $x = 1$). Hence, the domain of $(f \circ g)$ is the set of nonnegative reals except for 1. Expressed using intervals, that's $[0, 1) \cup (1, \infty)$.

7. (Bonus Problem)
Suppose that f and g are one-to-one functions. Express $(f \circ g)^{-1}$ (i.e., the inverse of $(f \circ g)$) as the composition of two functions. (Hint: both f and g are one-to-one and hence both have inverses.)

Solution: To apply $(f \circ g)$, one first applies g and then f. To invert that, you would first undo the effects of applying f and then undo the effects of applying g. In other words, you would first apply f^{-1} and then apply g^{-1}. But that corresponds to applying $(g^{-1} \circ f^{-1})$!

That is, we are claiming that $(f \circ g)^{-1} = (g^{-1} \circ f^{-1})$.

To verify:

$$((g^{-1} \circ f^{-1}) \circ (f \circ g))(x) = g^{-1}(f^{-1}(f(g(x)))) = g^{-1}(g(x)) = x$$