THEOREMS FROM GRIES AND SCHNEIDER'S LADM

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ABSTRACT. This is a collection of the axioms and theorems in Gries and Schneider's book *A Logical Approach to Discrete Math* (LADM), Springer-Verlag, 1993. The numbering is consistent with that text. Additional theorems not included or numbered in LADM are indicated by a three-part number. This document serves as a reference for homework exercises and taking exams.

TABLE OF PRECEDENCES

(a) $[x := e]$ (textual substitution) (highest precedence)
(b) . (function application)
(c) unary prefix operators: $+ \quad - \quad \neg \quad \# \quad \sim \quad \mathcal{P}$
(d) $**$
(e) $\cdot \quad / \quad \div$ **mod** **gcd**
(f) $+ \quad - \quad \cup \quad \cap \quad \times \quad \circ \quad \bullet$
(g) $\downarrow \quad \uparrow$
(h) $\#$
(i) $\triangleleft \quad \triangleright \quad \hat{}$
(j) $= \quad < \quad > \quad \in \quad \subset \quad \subseteq \quad \supset \quad \supseteq \quad |$ (conjunctional, see page 29)
(k) $\vee \quad \wedge$
(l) $\Rightarrow \quad \Leftarrow$
(m) $\equiv$

All nonassociative binary infix operators associate from left to right except $**$, $\triangleleft$, and $\Rightarrow$, which associate from right to left.

Definition of $/$: The operators on lines (j), (l), and (m) may have a slash $/$ through them to denote negation—e.g. $x \notin T$ is an abbreviation for $\neg(x \in T)$.

SOME BASIC TYPES

Name	Symbol	Type (set of values)
integer	$\mathbb{Z}$	integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
nat	$\mathbb{N}$	natural numbers: $0, 1, 2, \ldots$
positive	$\mathbb{Z}^+$	positive integers: $1, 2, 3, \ldots$
negative	$\mathbb{Z}^-$	negative integers: $-1, -2, -3, \ldots$
rational	$\mathbb{Q}$	rational numbers: i/j for i, j integers, $j \neq 0$
reals	$\mathbb{R}$	real numbers
positive reals	$\mathbb{R}^+$	positive real numbers
bool	$\mathbb{B}$	booleans: $true, false$

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Theorems of the Propositional Calculus

Equivalence and $true$.

(3.1) **Axiom, Associativity of $\equiv$:** $((p \equiv q) \equiv r) \;\equiv\; (p \equiv (q \equiv r))$

(3.2) **Axiom, Symmetry of $\equiv$:** $p \equiv q \equiv q \equiv p$

(3.3) **Axiom, Identity of $\equiv$:** $true \equiv q \equiv q$

(3.4) $true$

(3.5) **Reflexivity of $\equiv$:** $p \equiv p$

Negation, inequivalence, and $false$.

(3.8) **Definition of $false$:** $false \equiv \neg true$

(3.9) **Axiom, Distributivity of $\neg$ over $\equiv$:** $\neg(p \equiv q) \;\equiv\; \neg p \equiv q$

(3.10) **Definition of $\not\equiv$:** $(p \not\equiv q) \;\equiv\; \neg(p \equiv q)$

(3.11) $\neg p \equiv q \equiv p \equiv \neg q$

(3.12) **Double negation:** $\neg\neg p \equiv p$

(3.13) **Negation of $false$:** $\neg false \equiv true$

(3.14) $(p \not\equiv q) \;\equiv\; \neg p \equiv q$

(3.15) $\neg p \equiv p \equiv false$

(3.16) **Symmetry of $\not\equiv$:** $(p \not\equiv q) \;\equiv\; (q \not\equiv p)$

(3.17) **Associativity of $\not\equiv$:** $((p \not\equiv q) \not\equiv r) \;\equiv\; (p \not\equiv (q \not\equiv r))$

(3.18) **Mutual associativity:** $((p \not\equiv q) \equiv r) \;\equiv\; (p \not\equiv (q \equiv r))$

(3.19) **Mutual interchangeability:** $p \not\equiv q \equiv r \quad\equiv\quad p \equiv q \not\equiv r$

(3.19.1) $p \not\equiv p \not\equiv q \;\equiv\; q$

Disjunction.

(3.24) **Axiom, Symmetry of $\vee$:** $p \vee q \equiv q \vee p$

(3.25) **Axiom, Associativity of $\vee$:** $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(3.26) **Axiom, Idempotency of $\vee$:** $p \vee p \equiv p$

(3.27) **Axiom, Distributivity of $\vee$ over $\equiv$:** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$

(3.28) **Axiom, Excluded middle:** $p \vee \neg p$

(3.29) **Zero of $\vee$:** $p \vee true \equiv true$

(3.30) **Identity of $\vee$:** $p \vee false \equiv p$

(3.31) **Distributivity of $\vee$ over $\vee$:** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(3.32) $p \vee q \equiv p \vee \neg q \equiv p$

Conjunction.

(3.35) **Axiom, Golden rule:** $p \wedge q \;\equiv\; p \;\equiv\; q \;\equiv\; p \vee q$

(3.36) **Symmetry of $\wedge$:** $p \wedge q \equiv q \wedge p$

(3.37) **Associativity of $\wedge$:** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(3.38) **Idempotency of $\wedge$:** $p \wedge p \equiv p$

(3.39) **Identity of $\wedge$:** $p \wedge true \equiv p$

(3.40) **Zero of $\wedge$:** $p \wedge false \equiv false$

(3.41) **Distributivity of $\wedge$ over $\wedge$:** $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(3.42) **Contradiction:** $p \wedge \neg p \equiv false$

(3.43) **Absorption:**
(a) $p \wedge (p \vee q) \equiv p$
(b) $p \vee (p \wedge q) \equiv p$

(3.44) **Absorption:**
(a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
(b) $p \vee (\neg p \wedge q) \equiv p \vee q$

(3.45) **Distributivity of $\vee$ over $\wedge$:** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(3.46) **Distributivity of $\wedge$ over $\vee$:** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

(3.47) **De Morgan:**
(a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
(b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$

(3.48) $p \wedge q \ \equiv \ p \wedge \neg q \ \equiv \ \neg p$

(3.49) $p \wedge (q \equiv r) \ \equiv \ p \wedge q \ \equiv \ p \wedge r \ \equiv \ p$

(3.50) $p \wedge (q \equiv p) \ \equiv \ p \wedge q$

(3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \ \equiv \ (p \equiv q) \wedge (r \equiv q)$

(3.52) **Equivalence:** $p \equiv q \ \equiv \ (p \wedge q) \vee (\neg p \wedge \neg q)$

(3.53) **Exclusive or:** $p \not\equiv q \ \equiv \ (\neg p \wedge q) \vee (p \wedge \neg q)$

(3.55) $(p \wedge q) \wedge r \ \equiv \ p \ \equiv \ q \ \equiv \ r \ \equiv \ p \vee q \ \equiv \ q \vee r \ \equiv \ r \vee p \ \equiv \ p \vee q \vee r$

Implication.

(3.57) **Definition of Implication:** $p \Rightarrow q \ \equiv \ p \vee q \ \equiv \ q$

(3.58) **Axiom, Consequence:** $p \Leftarrow q \ \equiv \ q \Rightarrow p$

(3.59) **Implication:** $p \Rightarrow q \ \equiv \ \neg p \vee q$

(3.60) **Implication:** $p \Rightarrow q \ \equiv \ p \wedge q \ \equiv \ p$

(3.61) **Contrapositive:** $p \Rightarrow q \ \equiv \ \neg q \Rightarrow \neg p$

(3.62) $p \Rightarrow (q \equiv r) \ \equiv \ p \wedge q \ \equiv \ p \wedge r$

(3.63) **Distributivity of $\Rightarrow$ over $\equiv$:** $p \Rightarrow (q \equiv r) \ \equiv \ (p \Rightarrow q) \ \equiv \ (p \Rightarrow r)$

(3.63.1) **Distributivity of $\Rightarrow$ over $\wedge$:** $p \Rightarrow q \wedge r \ \equiv \ (p \Rightarrow q) \wedge (p \Rightarrow r)$

(3.63.2) **Distributivity of $\Rightarrow$ over $\vee$:** $p \Rightarrow q \vee r \ \equiv \ (p \Rightarrow q) \vee (p \Rightarrow r)$

(3.64) $p \Rightarrow (q \Rightarrow r) \ \equiv \ (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$

(3.65) **Shunting:** $p \wedge q \Rightarrow r \ \equiv \ p \Rightarrow (q \Rightarrow r)$

(3.66) $p \wedge (p \Rightarrow q) \ \equiv \ p \wedge q$

(3.67) $p \wedge (q \Rightarrow p) \ \equiv \ p$

(3.68) $p \vee (p \Rightarrow q) \ \equiv \ true$

(3.69) $p \vee (q \Rightarrow p) \ \equiv \ q \Rightarrow p$

(3.70) $p \vee q \Rightarrow p \wedge q \ \equiv \ p \ \equiv \ q$

(3.71) **Reflexivity of $\Rightarrow$:** $p \Rightarrow p$

(3.72) **Right zero of $\Rightarrow$:** $p \Rightarrow true \ \equiv \ true$

(3.73) **Left identity of $\Rightarrow$:** $true \Rightarrow p \ \equiv \ p$

(3.74) $p \Rightarrow false \ \equiv \ \neg p$

(3.74.1) $\neg p \Rightarrow false \;\equiv\; p$

(3.75) $false \Rightarrow p \;\equiv\; true$

(3.76) **Weakening/strengthening:**

(a) $p \Rightarrow p \vee q$ (Weakening the consequent)

(b) $p \wedge q \Rightarrow p$ (Strengthening the antecedent)

(c) $p \wedge q \Rightarrow p \vee q$ (Weakening/strengthening)

(d) $p \vee (q \wedge r) \Rightarrow p \vee q$

(e) $p \wedge q \Rightarrow p \wedge (q \vee r)$

(3.76.1) $p \wedge q \Rightarrow p \vee r$ (Weakening/strengthening)

(3.76.2) $(p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$

(3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$

(3.77.1) **Modus tollens:** $(p \Rightarrow q) \wedge \neg q \Rightarrow \neg p$

(3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \;\equiv\; (p \vee q \Rightarrow r)$

(3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \;\equiv\; r$

(3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \;\equiv\; (p \equiv q)$

(3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$

(3.82) **Transitivity:**

(a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$

(c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

(3.82.1) **Transitivity of $\equiv$:** $(p \equiv q) \wedge (q \equiv r) \Rightarrow (p \equiv r)$

(3.82.2) $(p \equiv q) \Rightarrow (p \Rightarrow q)$

Leibniz as an axiom.

This section uses the following notation: E^z_X means $E[z := X]$.

(3.83) **Axiom, Leibniz:** $e = f \Rightarrow E^z_e = E^z_f$

(3.84) **Substitution:**

(a) $(e = f) \wedge E^z_e \;\equiv\; (e = f) \wedge E^z_f$

(b) $(e = f) \Rightarrow E^z_e \;\equiv\; (e = f) \Rightarrow E^z_f$

(c) $q \wedge (e = f) \Rightarrow E^z_e \;\equiv\; q \wedge (e = f) \Rightarrow E^z_f$

(3.85) **Replace by $true$:**

(a) $p \Rightarrow E^z_p \;\equiv\; p \Rightarrow E^z_{true}$

(b) $q \wedge p \Rightarrow E^z_p \;\equiv\; q \wedge p \Rightarrow E^z_{true}$

(3.86) **Replace by $false$:**

(a) $E^z_p \Rightarrow p \;\equiv\; E^z_{false} \Rightarrow p$

(b) $E^z_p \Rightarrow p \vee q \;\equiv\; E^z_{false} \Rightarrow p \vee q$

(3.87) **Replace by $true$:** $p \wedge E^z_p \;\equiv\; p \wedge E^z_{true}$

(3.88) **Replace by $false$:** $p \vee E^z_p \;\equiv\; p \vee E^z_{false}$

(3.89) **Shannon:** $E^z_p \;\equiv\; (p \wedge E^z_{true}) \vee (\neg p \wedge E^z_{false})$

(3.89.1) $E^z_{true} \wedge E^z_{false} \Rightarrow E^z_p$

Additional theorems concerning implication.

(4.1) $p \Rightarrow (q \Rightarrow p)$

(4.2) **Monotonicity of $\vee$:** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$

(4.3) **Monotonicity of $\wedge$:** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

Proof technique metatheorems.

(4.4) **Deduction (assume conjuncts of antecedent):**
To prove $P_1 \wedge P_2 \Rightarrow Q$, assume P_1 and P_2, and prove Q.
You cannot use textual substitution in P_1 or P_2.

(4.5) **Case analysis:** If E^z_{true} and E^z_{false} are theorems, then so is E^z_P.

(4.6) **Case analysis:** $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$

(4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.

(4.7.1) **Truth implication:** To prove P, prove $true \Rightarrow P$.

(4.9) **Proof by contradiction:** To prove P, prove $\neg P \Rightarrow false$.

(4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$.

General Laws of Quantification

For symmetric and associative binary operator $\star$ with identity u.

(8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$

(8.14) **Axiom, One-point rule:** Provided $\neg occurs(`x', `E')$,
$(\star x \mid x = E : P) = P[x := E]$

(8.15) **Axiom, Distributivity:** Provided $P, Q : \mathbb{B}$ or R is finite,
$(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$

(8.16) **Axiom, Range split:** Provided $R \wedge S \equiv false$ and $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.17) **Axiom, Range split:** Provided $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.18) **Axiom, Range split for idempotent $\star$:** Provided $P : \mathbb{B}$ or R and S are finite,
$(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$

(8.19) **Axiom, Interchange of dummies:** Provided $\star$ is idempotent or R and Q are finite,
$\neg occurs(`y', `R')$, $\neg occurs(`x', `Q')$,
$(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$

(8.20) **Axiom, nesting:** Provided $\neg occurs(`y', `R')$,
$(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$

(8.21) **Axiom, Dummy renaming:** Provided $\neg occurs(`y', `R, P')$,
$(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$

(8.22) **Change of dummy:** Provided $\neg occurs(`y', `R, P')$, and f has an inverse,
$(\star x \mid R : P) = (\star y \mid R[x := f.y] : P[x := f.y])$

(8.23) **Split off term:** For n: $\mathbb{N}$,
(a) $(\star i \mid 0 \le i < n+1 : P) = (\star i \mid 0 \le i < n : P) \star P[i := n]$
(b) $(\star i \mid 0 \le i < n+1 : P) = P[i := 0] \star (\star i \mid 0 < i < n+1 : P)$

THEOREMS OF THE PREDICATE CALCULUS

Universal quantification.

Notation: $(\star x \mid : P)$ means $(\star x \mid true : P)$.

(9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid : R \Rightarrow P)$

(9.3) **Trading:**

(a) $(\forall x \mid R : P) \equiv (\forall x \mid : \neg R \vee P)$

(b) $(\forall x \mid R : P) \equiv (\forall x \mid : R \wedge P \equiv R)$

(c) $(\forall x \mid R : P) \equiv (\forall x \mid : R \vee P \equiv P)$

(9.4) **Trading:**

(a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$

(b) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$

(c) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$

(d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$

(9.4.1) **Universal double trading:** $(\forall x \mid R : P) \equiv (\forall x \mid \neg P : \neg R)$

(9.5) **Axiom, Distributivity of $\vee$ over $\forall$:** Provided $\neg occurs(\text{`}x\text{'}, \text{`}P\text{'})$,
$P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$

(9.6) Provided $\neg occurs(\text{`}x\text{'}, \text{`}P\text{'})$, $(\forall x \mid R : P) \equiv P \vee (\forall x \mid : \neg R)$

(9.7) **Distributivity of $\wedge$ over $\forall$:** Provided $\neg occurs(\text{`}x\text{'}, \text{`}P\text{'})$,
$\neg(\forall x \mid : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$

(9.8) $(\forall x \mid R : true) \equiv true$

(9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$

(9.10) **Range weakening/strengthening:** $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$

(9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$

(9.12) **Monotonicity of $\forall$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$

(9.13) **Instantiation:** $(\forall x \mid : P) \Rightarrow P[x := E]$

(9.16) **Metatheorem:** P is a theorem iff $(\forall x \mid : P)$ is a theorem.

Existential quantification.

(9.17) **Axiom, Generalized De Morgan:** $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$

(9.18) **Generalized De Morgan:**

(a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$

(b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$

(c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$

(9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid : R \wedge P)$

(9.20) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$

(9.20.1) **Existential double trading:** $(\exists x \mid R : P) \equiv (\exists x \mid P : R)$

(9.20.2) $(\exists x \mid : R) \Rightarrow ((\forall x \mid R : P) \Rightarrow (\exists x \mid R : P))$

(9.21) **Distributivity of $\wedge$ over $\exists$:** Provided $\neg occurs(\text{`}x\text{'}, \text{`}P\text{'})$,
$P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$

(9.22) Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$, $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid : R)$

(9.23) **Distributivity of $\vee$ over $\exists$:** Provided $\neg occurs(\text{'}x\text{'}, \text{'}P\text{'})$,
$(\exists x \mid : R) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$

(9.24) $(\exists x \mid R : false) \equiv false$

(9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$

(9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$

(9.27) **Monotonicity of $\exists$:** $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$

(9.28) **$\exists$-Introduction:** $P[x := E] \Rightarrow (\exists x \mid : P)$

(9.29) **Interchange of quantification:** Provided $\neg occurs(\text{'}y\text{'}, \text{'}R\text{'})$ and $\neg occurs(\text{'}x\text{'}, \text{'}Q\text{'})$,
$(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$

(9.30) Provided $\neg occurs(\text{'}\hat{x}\text{'}, \text{'}Q\text{'})$,
$(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem.

A Theory of Sets

(11.2) $\{e_0, \ldots, e_{n-1}\} = \{x \mid x = e_0 \vee \cdots \vee x = e_{n-1} : x\}$

(11.3) **Axiom, Set membership:** Provided $\neg occurs(\text{'}x\text{'}, \text{'}F\text{'})$,
$F \in \{x \mid R : E\} \equiv (\exists x \mid R : F = E)$

(11.4) **Axiom, Extensionality:** $S = T \ \equiv \ (\forall x \mid : x \in S \ \equiv \ x \in T)$

(11.4.1) **Axiom, Empty set:** $\emptyset = \{x \mid false : E\}$

(11.4.2) $e \in \emptyset \ \equiv \ false$

(11.4.3) **Axiom, Universe:** $\mathbf{U} = \{x \mid : x\}$, $\mathbf{U}$: $set(t) = \{x : t \mid : x\}$

(11.4.4) $e \in \mathbf{U} \ \equiv \ true$, for e: t and $\mathbf{U}$: $set(t)$

(11.5) $S = \{x \mid x \in S : x\}$

(11.5.1) **Axiom, Abbreviation:** For x a single variable, $\{x \mid R\} = \{x \mid R : x\}$

(11.6) Provided $\neg occurs(\text{'}y\text{'}, \text{'}R\text{'})$ and $\neg occurs(\text{'}y\text{'}, \text{'}E\text{'})$,
$\{x \mid R : E\} = \{y \mid (\exists x \mid R : y = E)\}$

(11.7) $x \in \{x \mid R\} \ \equiv \ R$
R is the characteristic predicate of the set.

(11.7.1) $y \in \{x \mid R\} \ \equiv \ R[x := y]$ for any expression y

(11.9) $\{x \mid Q\} = \{x \mid R\} \ \equiv \ (\forall x \mid : Q \equiv R)$

(11.10) $\{x \mid Q\} = \{x \mid R\}$ is valid iff $Q \equiv R$ is valid.

(11.11) **Methods for proving set equality $S = T$:**
(a) Use Leibniz directly.
(b) Use axiom Extensionality (11.4) and prove the (9.8) Lemma
$v \in S \ \equiv \ v \in T$ for an arbitrary value v.
(c) Prove $Q \equiv R$ and conclude $\{x \mid Q\} = \{x \mid R\}$.

Operations on sets.

(11.12) **Axiom, Size:** $\#S = (\Sigma x \mid x \in S : 1)$

(11.13) **Axiom, Subset:** $S \subseteq T \equiv (\forall x \mid x \in S : x \in T)$

(11.14) **Axiom, Proper subset:** $S \subset T \ \equiv \ S \subseteq T \wedge S \neq T$

(11.15) **Axiom, Superset:** $T \supseteq S \equiv S \subseteq T$

(11.16) **Axiom, Proper superset:** $T \supset S \equiv S \subset T$

(11.17) **Axiom, Complement:** $v \in \sim S \equiv v \in \mathbf{U} \wedge v \notin S$

(11.18) $v \in \sim S \equiv v \notin S$, for v in $\mathbf{U}$

(11.19) $\sim\sim S = S$

(11.20) **Axiom, Union:** $v \in S \cup T \equiv v \in S \vee v \in T$

(11.21) **Axiom, Intersection:** $v \in S \cap T \equiv v \in S \wedge v \in T$

(11.22) **Axiom, Difference:** $v \in S - T \equiv v \in S \wedge v \notin T$

(11.23) **Axiom, Power set:** $v \in \mathcal{P}S \equiv v \subseteq S$

(11.24) **Definition.** Let E_s be a set expression constructed from set variables, $\emptyset$, $\mathbf{U}$, $\sim$, $\cup$, and $\cap$.
Then E_p is the expression constructed from E_s by replacing:
$\emptyset$ with $false$, $\mathbf{U}$ with $true$, $\cup$ with $\vee$, $\cap$ with $\wedge$, $\sim$ with $\neg$.
The construction is reversible: E_s can be constructed from E_p.

(11.25) **Metatheorem.** For any set expressions E_s and F_s:
(a) $E_s = F_s$ is valid iff $E_p \equiv F_p$ is valid,
(b) $E_s \subseteq F_s$ is valid iff $E_p \Rightarrow F_p$ is valid,
(c) $E_s = \mathbf{U}$ is valid iff E_p is valid.

Basic properties of $\cup$.

(11.26) **Symmetry of $\cup$:** $S \cup T = T \cup S$

(11.27) **Associativity of $\cup$:** $(S \cup T) \cup U = S \cup (T \cup U)$

(11.28) **Idempotency of $\cup$:** $S \cup S = S$

(11.29) **Zero of $\cup$:** $S \cup \mathbf{U} = \mathbf{U}$

(11.30) **Identity of $\cup$:** $S \cup \emptyset = S$

(11.31) **Weakening:** $S \subseteq S \cup T$

(11.32) **Excluded middle:** $S \cup \sim S = \mathbf{U}$

Basic properties of $\cap$.

(11.33) **Symmetry of $\cap$:** $S \cap T = T \cap S$

(11.34) **Associativity of $\cap$:** $(S \cap T) \cap U = S \cap (T \cap U)$

(11.35) **Idempotency of $\cap$:** $S \cap S = S$

(11.36) **Zero of $\cap$:** $S \cap \emptyset = \emptyset$

(11.37) **Identity of $\cap$:** $S \cap \mathbf{U} = S$

(11.38) **Strengthening:** $S \cap T \subseteq S$

(11.39) **Contradiction:** $S \cap \sim S = \emptyset$

Basic properties of combinations of $\cup$ and $\cap$.

(11.40) **Distributivity of $\cup$ over $\cap$:** $S \cup (T \cap U) = (S \cup T) \cap (S \cup U)$

(11.41) **Distributivity of $\cap$ over $\cup$:** $S \cap (T \cup U) = (S \cap T) \cup (S \cap U)$

(11.42) **De Morgan:**

(a) $\sim (S \cup T) = \sim S \cap \sim T$

(b) $\sim (S \cap T) = \sim S \cup \sim T$

Additional properties of $\cup$ and $\cap$.

(11.43) $S \subseteq T \wedge U \subseteq V \Rightarrow (S \cup U) \subseteq (T \cup V)$

(11.44) $S \subseteq T \wedge U \subseteq V \Rightarrow (S \cap U) \subseteq (T \cap V)$

(11.45) $S \subseteq T \equiv S \cup T = T$

(11.46) $S \subseteq T \equiv S \cap T = S$

(11.47) $S \cup T = \mathbf{U} \ \equiv \ (\forall x \mid x \in \mathbf{U} : x \notin S \Rightarrow x \in T)$

(11.48) $S \cap T = \emptyset \ \equiv \ (\forall x \mid : x \in S \Rightarrow x \notin T)$

Properties of set difference.

(11.49) $S - T = S \cap \sim T$

(11.50) $S - T \subseteq S$

(11.51) $S - \emptyset = S$

(11.52) $S \cap (T - S) = \emptyset$

(11.53) $S \cup (T - S) = S \cup T$

(11.54) $S - (T \cup U) = (S - T) \cap (S - U)$

(11.55) $S - (T \cap U) = (S - T) \cup (S - U)$

Implication versus subset.

(11.56) $(\forall x \mid : P \Rightarrow Q) \ \equiv \ \{x \mid P\} \subseteq \{x \mid Q\}$

Properties of subset.

(11.57) **Antisymmetry:** $S \subseteq T \wedge T \subseteq S \ \equiv \ S = T$

(11.58) **Reflexivity:** $S \subseteq S$

(11.59) **Transitivity:** $S \subseteq T \wedge T \subseteq U \Rightarrow S \subseteq U$

(11.60) $\emptyset \subseteq S$

(11.61) $S \subset T \ \equiv \ S \subseteq T \wedge \neg(T \subseteq S)$

(11.62) $S \subset T \ \equiv \ S \subseteq T \wedge (\exists x \mid x \in T : x \notin S)$

(11.63) $S \subseteq T \ \equiv \ S \subset T \vee S = T$

(11.64) $S \not\subset S$

(11.65) $S \subset T \Rightarrow S \subseteq T$

(11.66) $S \subset T \Rightarrow T \not\subseteq S$

(11.67) $S \subseteq T \Rightarrow T \not\subset S$

(11.68) $S \subseteq T \wedge \neg(U \subseteq T) \Rightarrow \neg(U \subseteq S)$

(11.69) $(\exists x \mid x \in S : x \notin T) \Rightarrow S \neq T$

(11.70) **Transitivity:**
(a) $S \subseteq T \wedge T \subset U \Rightarrow S \subset U$
(b) $S \subset T \wedge T \subseteq U \Rightarrow S \subset U$
(c) $S \subset T \wedge T \subset U \Rightarrow S \subset U$

Theorems concerning power set $\mathcal{P}$.

(11.71) $\mathcal{P}\emptyset = \{\emptyset\}$

(11.72) $S \in \mathcal{P}S$

(11.73) $\#(\mathcal{P}S) = 2^{\#S}$ (for finite set S)

(11.76) **Axiom, Partition:** Set S partitions T if
(i) the sets in S are pairwise disjoint and
(ii) the union of the sets in S is T, that is, if
$(\forall u, v \mid u \in S \wedge v \in S \wedge u \neq v : u \cap v = \emptyset) \wedge (\cup u \mid u \in S : u) = T$

Bags.

(11.79) **Axiom, Membership:** $v \in \{\!\mid x \mid R : E \mid\!\} \equiv (\exists x \mid R : v = E)$

(11.80) **Axiom, Size:** $\#\{\!\mid x \mid R : E \mid\!\} = (\Sigma x \mid R : 1)$

(11.81) **Axiom, Number of occurrences:** $v\#\{\!\mid x \mid R : E \mid\!\} = (\Sigma x \mid R \wedge v = E : 1)$

(11.82) **Axiom, Bag equality:** $B = C \equiv (\forall v \mid : v\#B = v\#C)$

(11.83) **Axiom, Subbag:** $B \subseteq C \equiv (\forall v \mid : v\#B \leq v\#C)$

(11.84) **Axiom, Proper subbag:** $B \subset C \equiv B \subseteq C \wedge B \neq C$

(11.85) **Axiom, Union:** $B \cup C = \{\!\mid v, i \mid 0 \leq i < v\#B + v\#C : v \mid\!\}$

(11.86) **Axiom, Intersection:** $B \cap C = \{\!\mid v, i \mid 0 \leq i < v\#B \downarrow v\#C : v \mid\!\}$

(11.87) **Axiom, Difference:** $B - C = \{\!\mid v, i \mid 0 \leq i < v\#B - v\#C : v \mid\!\}$

MATHEMATICAL INDUCTION

(12.3) **Axiom, Mathematical Induction over $\mathbb{N}$:**
$(\forall n{:}\ \mathbb{N} \mid : (\forall i \mid 0 \leq i < n : P.i) \Rightarrow P.n) \Rightarrow (\forall n{:}\ \mathbb{N} \mid : P.n)$

(12.4) **Mathematical Induction over $\mathbb{N}$:**
$(\forall n{:}\ \mathbb{N} \mid : (\forall i \mid 0 \leq i < n : P.i) \Rightarrow P.n) \equiv (\forall n{:}\ \mathbb{N} \mid : P.n)$

(12.5) **Mathematical Induction over $\mathbb{N}$:**
$P.0 \wedge (\forall n{:}\ \mathbb{N} \mid : (\forall i \mid 0 \leq i \leq n : P.i) \Rightarrow P(n+1)) \equiv (\forall n{:}\ \mathbb{N} \mid : P.n)$

(12.11) **Definition, b to the power n:**
$b^0 = 1$
$b^{n+1} = b \cdot b^n$ for $n \geq 0$

(12.12) b **to the power** n**:**
$b^0 = 1$
$b^n = b \cdot b^{n-1}$ for $n \geq 1$

(12.13) **Definition, factorial:**
$0! = 1$
$n! = n \cdot (n-1)!$ for $n > 0$

(12.14) **Definition, Fibonacci:**
$F_0 = 0, \quad F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$ for $n > 1$

(12.14.1) **Definition, Golden Ratio:** $\phi = (1+\sqrt{5})/2 \approx 1.618 \qquad \hat{\phi} = (1-\sqrt{5})/2 \approx -0.618$

(12.15) $\phi^2 = \phi + 1$ and $\hat{\phi}^2 = \hat{\phi} + 1$

(12.16) $F_n \leq \phi^{n-1}$ for $n \geq 1$

(12.16.1) $\phi^{n-2} \leq F_n$ for $n \geq 1$

(12.17) $F_{n+m} = F_m \cdot F_{n+1} + F_{m-1} \cdot F_n$ for $n \geq 0$ and $m \geq 1$

Inductively defined binary trees.

(12.30) **Definition, Binary Tree:**
$\emptyset$ is a binary tree, called the empty tree.
(d, l, r) is a binary tree, for d: $\mathbb{Z}$ and l, r binary trees.

(12.31) **Definition, Number of Nodes:**
$\#\emptyset = 0$
$\#(d, l, r) = 1 + \#l + \#r$

(12.32) **Definition, Height:**
$height.\emptyset = 0$
$height.(d, l, r) = 1 + max(height.l, height.r)$

(12.32.1) **Definition, Leaf:** A leaf is a node with no children (i.e. two empty subtrees).

(12.32.2) **Definition, Internal node:** An internal node is a node that is not a leaf.

(12.32.3) **Definition, Complete:** A binary tree is complete if every node has either 0 or 2 children.

(12.33) The maximum number of nodes in a tree with height n is $2^n - 1$ for $n \geq 0$.

(12.34) The minimum number of nodes in a tree with height n is n for $n \geq 0$.

(12.35) (a) The maximum number of leaves in a tree with height n is 2^{n-1} for $n > 0$.
(b) The maximum number of internal nodes in a tree with height n is $2^{n-1} - 1$ for $n > 0$.

(12.36) (a) The minimum number of leaves in a tree with height n is 1 for $n > 0$.
(b) The minimum number of internal nodes in a tree with height n is $n - 1$ for $n > 0$.

(12.37) Every nonempty complete tree has an odd number of nodes.

A THEORY OF PROGRAMS

(p.1) **Axiom, Excluded miracle:** $wp.S.\mathit{false} \equiv \mathit{false}$

(p.2) **Axiom, Conjunctivity:** $wp.S.(X \wedge Y) \equiv wp.S.X \wedge wp.S.Y$

(p.3) **Monotonicity:** $(X \Rightarrow Y) \Rightarrow (wp.S.X \Rightarrow wp.S.Y)$

(p.4) **Definition, Hoare triple:** $\{Q\}\ S\ \{R\} \equiv Q \Rightarrow wp.S.R$

(p.4.1) $\{wp.S.R\}\ S\ \{R\}$

(p.5) **Postcondition rule:** $\{Q\}\ S\ \{A\} \wedge (A \Rightarrow R) \ \Rightarrow\ \{Q\}\ S\ \{R\}$

(p.6) **Definition, Program equivalence:** $S = T \quad \equiv \quad$ (For all $R, wp.S.R \equiv wp.T.R$)

(p.7) $(Q \Rightarrow A) \wedge \{A\}\ S\ \{R\} \ \Rightarrow\ \{Q\}\ S\ \{R\}$

(p.8) $\{Q0\}\ S\ \{R0\}\ \wedge \{Q1\}\ S\ \{R1\} \ \Rightarrow\ \{Q0 \wedge Q1\}\ S\ \{R0 \wedge R1\}$

(p.9) $\{Q0\}\ S\ \{R0\}\ \wedge \{Q1\}\ S\ \{R1\} \ \Rightarrow\ \{Q0 \vee Q1\}\ S\ \{R0 \vee R1\}$

(p.10) **Definition, skip:** $wp.skip.R \ \equiv\ R$

(p.11) $\{Q\}$ *skip* $\{R\} \quad \equiv \quad Q \Rightarrow R$

(p.12) **Definition, abort:** $wp.abort.R \ \equiv\ $ *false*

(p.13) $\{Q\}$ *abort* $\{R\} \quad \equiv \quad Q \equiv$ *false*

(p.14) **Definition, Composition:** $wp.(S;T).R \ \equiv\ wp.S.(wp.T.R)$

(p.15) $\{Q\}\ S\ \{H\} \quad \wedge \quad \{H\}\ T\ \{R\} \quad \Rightarrow \quad \{Q\}\ S;T\ \{R\}$

(p.16) **Identity of composition:**
(a) S ; *skip* $= S$ (b) *skip* ; $S\ = S$

(p.17) **Zero of composition:**
(a) S ; *abort* = *abort* (b) *abort* ; S = *abort*

(p.18) **Definition, Assignment:** $wp.(x := E).R \ \equiv\ R[x := E]$

(p.19) **Proof method for assignment:** (p.19) is (10.2)
To show that $x := E$ is an implementation of $\{Q\}x :=?\{R\}$,
prove $Q \Rightarrow R[x := E]$.

(p.20) $(x := x) =$ *skip*

(p.21) IFG : (p.21) is (10.6)
if $B1 \rightarrow S1$
[] $B2 \rightarrow S2$
[] $B3 \rightarrow S3$
fi

(p.22) **Definition,** IFG**:** $wp.IFG.R \quad \equiv \quad (B1 \vee B2 \vee B3) \quad \wedge$
$B1 \Rightarrow wp.S1.R \quad \wedge \quad B2 \Rightarrow wp.S2.R \quad \wedge \quad B3 \Rightarrow wp.S3.R$

(p.23) **Empty guard: if fi =** *abort*

(p.24) **Proof method for** IFG**:** (p.24) is (10.7)
To prove $\{Q\}IFG\{R\}$, it suffices to prove
(a) $Q \Rightarrow B1 \vee B2 \vee B3$,
(b) $\{Q \wedge B1\}\ S1\ \{R\}$,
(c) $\{Q \wedge B2\}\ S2\ \{R\}$, and
(d) $\{Q \wedge B3\}\ S3\ \{R\}$.

(p.25) $\neg(B1 \vee B2 \vee B3) \Rightarrow\ IFG\ =$ *abort*

(p.26) **One-guard rule:** $\{Q\}$ **if** $B \rightarrow S$ **fi** $\{R\} \ \Rightarrow\ \{Q\}\ S\ \{R\}$

(p.27) **Distributivity of program over alternation:**
if $B1 \rightarrow S1;T$ [] $B2 \rightarrow S2;T$ **fi** $=$ **if** $B1 \rightarrow S1$ [] $B2 \rightarrow S2$ **fi** ; T

(p.28) DO : **do** $B \rightarrow S$ **od**

(p.29) **Fundamental Invariance Theorem.** (p.29) is (12.43)

Suppose

- $\{P \wedge B\}\ S\ \{P\}$ holds—i.e. execution of S begun in a state in which P and B are $true$ terminates with P $true$—and
- $\{P\}$ **do** $B \rightarrow S$ **od** $\{true\}$—i.e. execution of the loop begun in a state in which P is $true$ terminates.

Then $\{P\}$ **do** $B \rightarrow S$ **od** $\{P \wedge \neg B\}$ holds.

(p.30) **Proof method for DO:** (p.30) is (12.45)

To prove $\{Q\}\ initialization;\ \{P\}$ **do** $B \rightarrow S$ **od** $\{R\}$,
it suffices to prove

(a) P is $true$ before execution of the loop: $\{Q\}\ initialization;\ \{P\}$,
(b) P is a loop invariant: $\{P \wedge B\}\ S\ \{P\}$,
(c) Execution of the loop terminates, and
(d) R holds upon termination: $P \wedge \neg B \Rightarrow R$.

(p.31) **False guard:** **do** $false \rightarrow S$ **od** = *skip*

RELATIONS AND FUNCTIONS

(14.2) **Axiom, Pair equality:** $\langle b, c\rangle = \langle b', c'\rangle \ \equiv\ b = b' \wedge c = c'$

(14.2.1) **Ordered pair one-point rule:** Provided $\neg occurs$(‘x, y’, ‘E, F’),
$(\star x, y \mid \langle x, y\rangle = \langle E, F\rangle : P) = P[x, y := E, F]$

(14.3) **Axiom, Cross product:** $S \times T = \{b, c \mid b \in S \wedge c \in T : \langle b, c\rangle\}$

(14.3.1) **Axiom, Ordered pair extensionality:**
$U = V \ \equiv\ (\forall x, y \mid : \langle x, y\rangle \in U \equiv \langle x, y\rangle \in V)$

Theorems for cross product.

(14.4) **Membership:** $\langle x, y\rangle \in S \times T \ \equiv\ x \in S \wedge y \in T$

(14.5) $\langle x, y\rangle \in S \times T \ \equiv\ \langle y, x\rangle \in T \times S$

(14.6) $S = \emptyset \Rightarrow S \times T = T \times S = \emptyset$

(14.7) $S \times T = T \times S \ \equiv\ S = \emptyset \ \vee\ T = \emptyset \ \vee\ S = T$

(14.8) **Distributivity of $\times$ over $\cup$:**
(a) $S \times (T \cup U) = (S \times T) \cup (S \times U)$
(b) $(S \cup T) \times U = (S \times U) \cup (T \times U)$

(14.9) **Distributivity of $\times$ over $\cap$:**
(a) $S \times (T \cap U) = (S \times T) \cap (S \times U)$
(b) $(S \cap T) \times U = (S \times U) \cap (T \times U)$

(14.10) **Distributivity of $\times$ over $-$:**
$S \times (T - U) = (S \times T) - (S \times U)$

(14.11) **Monotonicity:** $T \subseteq U \ \Rightarrow\ S \times T \subseteq S \times U$

(14.12) $S \subseteq U \wedge T \subseteq V \ \Rightarrow\ S \times T \subseteq U \times V$

(14.13) $S \times T \subseteq S \times U \wedge S \neq \emptyset \;\Rightarrow\; T \subseteq U$
(14.14) $(S \cap T) \times (U \cap V) = (S \times U) \cap (T \times V)$
(14.15) For finite S and T, $\#(S \times T) = \#S \cdot \#T$

Relations.

(14.15.1) **Definition, Binary relation:**
A *binary relation* over $B \times C$ is a subset of $B \times C$.
(14.15.2) **Definition, Identity:** The identity relation i_B on B is $i_B = \{x\colon B \mid\colon \langle x, x\rangle\}$
(14.15.3) **Identity lemma:** $\langle x, y\rangle \in i_B \;\equiv\; x = y$
(14.15.4) **Notation:** $\langle b, c\rangle \in \rho$ and $b\ \rho\ c$ are interchangeable notations.
(14.15.5) **Conjunctive meaning:** $b\ \rho\ c\ \sigma\ d \;\equiv\; b\ \rho\ c \;\wedge\; c\ \sigma\ d$

The *domain* $Dom.\rho$ and *range* $Ran.\rho$ of a relation ρ on $B \times C$ are defined by
(14.16) **Definition, Domain:** $Dom.\rho = \{b\colon B \mid (\exists c \mid\colon b\ \rho\ c)\}$
(14.17) **Definition, Range:** $Ran.\rho = \{c\colon C \mid (\exists b \mid\colon b\ \rho\ c)\}$

The $inverse$ ρ^{-1} of a relation ρ on $B \times C$ is the relation defined by
(14.18) **Definition, Inverse:** $\langle b, c\rangle \in \rho^{-1} \;\equiv\; \langle c, b\rangle \in \rho,$ for all $b\colon B$, $c\colon C$
(14.19) Let ρ and σ be relations.
(a) $Dom(\rho^{-1}) = Ran.\rho$
(b) $Ran(\rho^{-1}) = Dom.\rho$
(c) If ρ is a relation on $B \times C$, then ρ^{-1} is a relation on $C \times B$
(d) $(\rho^{-1})^{-1} = \rho$
(e) $\rho \subseteq \sigma \;\equiv\; \rho^{-1} \subseteq \sigma^{-1}$

Let ρ be a relation on $B \times C$ and σ be a relation on $C \times D$. The *product* of ρ and σ, denoted by $\rho \circ \sigma$, is the relation defined by
(14.20) **Definition, Product:** $\langle b, d\rangle \in \rho \circ \sigma \;\equiv\; (\exists c \mid c \in C : \langle b, c\rangle \in \rho \wedge \langle c, d\rangle \in \sigma)$
or, using the alternative notation by
(14.21) **Definition, Product:** $b\ (\rho \circ \sigma)\ d \;\equiv\; (\exists c \mid\colon b\ \rho\ c\ \sigma\ d)$

Theorems for relation product.

(14.22) **Associativity of $\circ$:** $\rho \circ (\sigma \circ \theta) = (\rho \circ \sigma) \circ \theta$
(14.23) **Distributivity of $\circ$ over $\cup$:**
(a) $\rho \circ (\sigma \cup \theta) = (\rho \circ \sigma) \cup (\rho \circ \theta)$
(b) $(\sigma \cup \theta) \circ \rho = (\sigma \circ \rho) \cup (\theta \circ \rho)$
(14.24) **Distributivity of $\circ$ over $\cap$:**
(a) $\rho \circ (\sigma \cap \theta) = (\rho \circ \sigma) \cap (\rho \circ \theta)$
(b) $(\sigma \cap \theta) \circ \rho = (\sigma \circ \rho) \cap (\theta \circ \rho)$

Theorems for powers of a relation.

(14.25) **Definition:**
$\rho^0 = i_B$
$\rho^{n+1} = \rho^n \circ \rho$ for $n \geq 0$

(14.26) $\rho^m \circ \rho^n = \rho^{m+n}$ for $m \geq 0, n \geq 0$

(14.27) $(\rho^m)^n = \rho^{m \cdot n}$ for $m \geq 0, n \geq 0$

(14.28) For ρ a relation on finite set B of n elements,
$(\exists i, j \mid 0 \leq i < j \leq 2^{n^2} : \rho^i = \rho^j)$

(14.29) Let ρ be a relation on a finite set B. Suppose $\rho^i = \rho^j$ and $0 \leq i < j$. Then
(a) $\rho^{i+k} = \rho^{j+k}$ for $k \geq 0$
(b) $\rho^i = \rho^{i+p\cdot(j-i)}$ for $p \geq 0$

Table 14.1 Classes of relations ρ over set B

	Name	Property	Alternative
(a)	reflexive	$(\forall b \mid : b \ \rho \ b)$	$i_B \subseteq \rho$
(b)	irreflexive	$(\forall b \mid : \neg(b \ \rho \ b))$	$i_B \cap \rho = \emptyset$
(c)	symmetric	$(\forall b, c \mid : b \ \rho \ c \ \equiv \ c \ \rho \ b)$	$\rho^{-1} = \rho$
(d)	antisymmetric	$(\forall b, c \mid : b \ \rho \ c \wedge c \ \rho \ b \Rightarrow b = c)$	$\rho \cap \rho^{-1} \subseteq i_B$
(e)	asymmetric	$(\forall b, c \mid : b \ \rho \ c \Rightarrow \neg(c \ \rho \ b))$	$\rho \cap \rho^{-1} = \emptyset$
(f)	transitive	$(\forall b, c, d \mid : b \ \rho \ c \wedge c \ \rho \ d \Rightarrow b \ \rho \ d)$	$\rho = (\cup i \mid i > 0 : \rho^i)$

(14.30.1) **Definition:** Let ρ be a relation on a set. The *reflexive closure* of ρ is the relation $r(\rho)$ that satisfies:
(a) $r(\rho)$ is reflexive;
(b) $\rho \subseteq r(\rho)$;
(c) If any relation σ is reflexive and $\rho \subseteq \sigma$, then $r(\rho) \subseteq \sigma$.

(14.30.2) **Definition:** Let ρ be a relation on a set. The *symmetric closure* of ρ is the relation $s(\rho)$ that satisfies:
(a) $s(\rho)$ is symmetric;
(b) $\rho \subseteq s(\rho)$;
(c) If any relation σ is symmetric and $\rho \subseteq \sigma$, then $s(\rho) \subseteq \sigma$.

(14.30.3) **Definition:** Let ρ be a relation on a set. The *transitive closure* of ρ is the relation ρ^+ that satisfies:
(a) ρ^+ is transitive;
(b) $\rho \subseteq \rho^+$;
(c) If any relation σ is transitive and $\rho \subseteq \sigma$, then $\rho^+ \subseteq \sigma$.

(14.30.4) **Definition:** Let ρ be a relation on a set. The *reflexive transitive closure* of ρ is the relation ρ^* that is both the reflexive and the transitive closure of ρ.

(14.31) (a) A reflexive relation is its own reflexive closure.
(b) A symmetric relation is its own symmetric closure.
(c) A transitive relation is its own transitive closure.

(14.32) Let ρ be a relation on a set B. Then,
(a) $r(\rho) = \rho \cup i_B$
(b) $s(\rho) = \rho \cup \rho^{-1}$
(c) $\rho^+ = (\cup i \mid 0 < i : \rho^i)$
(d) $\rho^* = \rho^+ \cup i_B$

Equivalence relations.

(14.33) **Definition:** A relation is an *equivalence relation* iff it is reflexive, symmetric, and transitive

(14.34) **Definition:** Let ρ be an equivalence relation on B. Then $[b]_\rho$, the *equivalence class* of b, is the subset of elements of B that are equivalent (under ρ) to b:
$x \in [b]_\rho \quad \equiv \quad x \; \rho \; b$

(14.35) Let ρ be an equivalence relation on B, and let b, c be members of B. The following three predicates are equivalent:
(a) $b \; \rho \; c$
(b) $[b] \cap [c] \neq \emptyset$
(c) $[b] = [c]$
That is, $(b \; \rho \; c) \; = \; ([b] \cap [c] \neq \emptyset) \; = \; ([b] = [c])$

(14.35.1) Let ρ be an equivalence relation on B. The equivalence classes partition B.

(14.36) Let P be the set of sets of a partition of B. The following relation ρ on B is an equivalence relation:
$b \; \rho \; c \equiv (\exists p \mid p \in P : b \in p \wedge c \in p)$

Functions.

(14.37) (a) **Definition:** A binary relation f on $B \times C$ is *determinate* iff
$(\forall b, c, c' \mid b \; f \; c \; \wedge \; b \; f \; c' : c = c')$
(b) **Definition:** A binary relation is a *function* iff it is determinate.

(14.37.1) **Notation:** $f.b = c$ and $b \; f \; c$ are interchangeable notations.

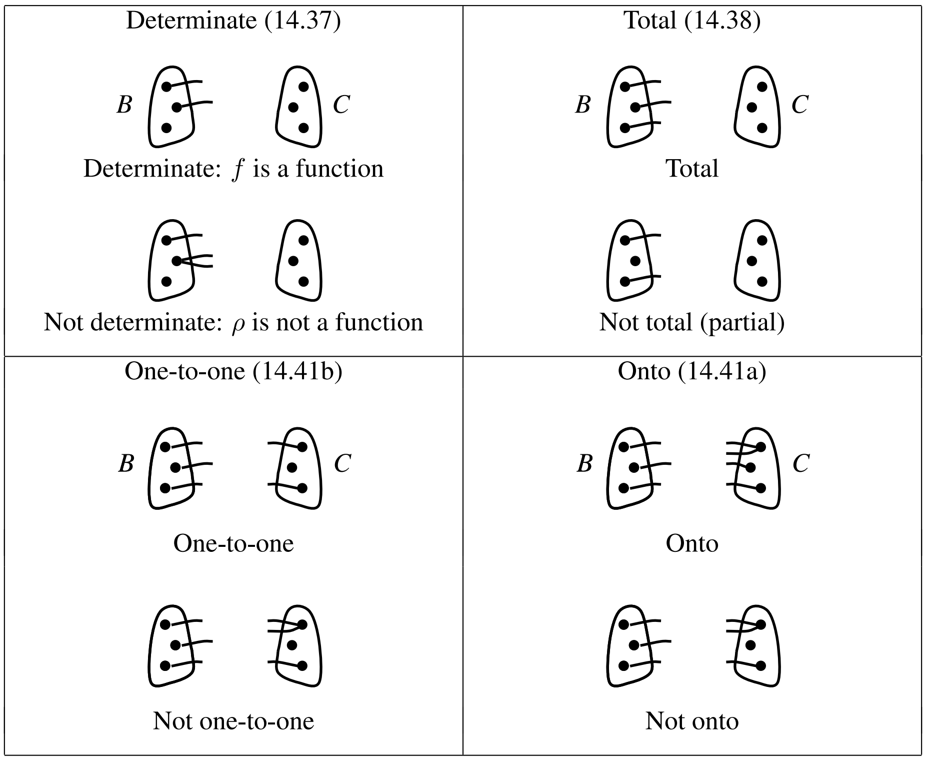
(14.38) **Definition:** A function f on $B \times C$ is *total* if $B = Dom.f$.
Otherwise it is *partial*.
We write $f : B \to C$ for the type of f if f is total and $f : B \leadsto C$ if f is partial.

(14.38.1) **Total:** A function f on $B \times C$ is total if, for an arbitrary element b: B,
$(\exists c\colon C \mid\colon f.b = c)$

(14.39) **Definition, Composition:** For functions f and g, $f \bullet g \; = \; g \circ f$.

(14.40) Let $g : B \to C$ and $f : C \to D$ be total functions.
Then the composition $f \bullet g$ of f and g is the total function defined by
$(f \bullet g).b = f(g.b)$

ρ a relation on $B \times C$
f a function, $f : B \to C$



Inverses of total functions.

(14.41) **Definitions:**

(a) Total function $f : B \to C$ is *onto* or *surjective* if $Ran.f = C$.

(b) Total function f is *one-to-one* or *injective* if
$(\forall b, b' \colon B, c \colon \mathsf{C} \mid \colon b\ f\ c \ \wedge\ b'\ f\ c \ \equiv\ b = b')$.

(c) Total function f is *bijective* if it is one-to-one and onto.

(14.42) Let f be a total function, and let f^{-1} be its relational inverse.

(a) Then f^{-1} is a function, i.e. is determinate, iff f is one-to-one.

(b) And, f^{-1} is total iff f is onto.

(14.43) **Definitions:** Let $f : B \to C$.

(a) A *left inverse* of f is a function $g : C \to B$ such that $g \bullet f = i_B$.

(b) A *right inverse* of f is a function $g : C \to B$ such that $f \bullet g = i_C$.

(c) Function g is an *inverse* of f if it is both a left inverse and a right inverse.

(14.44) Function $f : B \to C$ is onto iff f has a right inverse.

(14.45) Let $f : B \to C$ be total. Then f is one-to-one iff f has a left inverse.

(14.46) Let $f : B \to C$ be total. The following statements are equivalent.

(a) f is one-to-one and onto.

(b) There is a function $g : C \to B$ that is both a left and a right inverse of f.
(c) f has a left inverse and f has a right inverse.

Order relations.

(14.47) **Definition:** A binary relation ρ on a set B is called a *partial order on* b if it is reflexive, antisymmetric, and transitive. In this case, pair $\langle B, \rho\rangle$ is called a *partially ordered set* or *poset*.

We use the symbol $\preceq$ for an arbitrary partial order, sometimes writing $c \succeq b$ instead of $b \preceq c$.

(14.47.1) **Definition, Incomparable:** $incomp(b, c) \;\equiv\; \neg(b \preceq c) \wedge \neg(c \preceq b)$

(14.48) **Definition:** Relation $\prec$ is a *quasi order* or *strict partial order* if $\prec$ is transitive and irreflexive

(14.48.1) **Definition, Reflexive reduction:** Given $\preceq$, its *reflexive reduction* $\prec$ is computed by eliminating all pairs $\langle b, b\rangle$ from $\preceq$.

(14.48.2) Let $\prec$ be the reflexive reduction of $\preceq$. Then,
$\neg(b \preceq c) \;\equiv\; c \prec b \vee incomp(b, c)$

(14.49) (a) If ρ is a partial order over a set B, then $\rho - i_B$ is a quasi order.
(b) If ρ is a quasi order over a set B, then $\rho \cup i_B$ is a partial order.

Total orders and topological sort.

(14.50) **Definition:** A partial order $\preceq$ over B is called a *total* or *linear* order if $(\forall b, c \mid : b \preceq c \vee b \succeq c)$, i.e. iff $\preceq \cup \preceq^{-1} = B \times B$.
In this case, the pair $\langle B, \preceq\rangle$ is called a *linearly ordered set* or a *chain*.

(14.51) **Definitions:** Let S be a nonempty subset of poset $\langle U, \preceq\rangle$.
(a) Element b of S is a *minimal element of* S if no element of S is smaller than b, i.e. if $b \in S \wedge (\forall c \mid c \prec b : c \notin S)$.
(b) Element b of S is the *least element of* S if $b \in S \wedge (\forall c \mid c \in S : b \preceq c)$.
(c) Element b is a *lower bound of* S if $(\forall c \mid c \in S : b \preceq c)$.
(A lower bound of S need not be in S.)
(d) Element b is the *greatest lower bound of* S, written $glb.S$ if b is a lower bound and if every lower bound c satisfies $c \preceq b$.

(14.52) Every finite nonempty subset S of poset $\langle U, \preceq\rangle$ has a minimal element.

(14.53) Let B be a nonempty subset of poset $\langle U, \preceq\rangle$.
(a) A least element of B is also a minimal element of B (but not necessarily vice versa).
(b) A least element of B is also a greatest lower bound of B (but not necessarily vice versa).
(c) A lower bound of B that belongs to B is also a least element of B.

((14.54) **Definitions:** Let S be a nonempty subset of poset $\langle U, \preceq \rangle$.

(a) Element b of S is a *maximal element of* S if no element of S is larger than b, i.e. if $b \in S \wedge (\forall c \mid b \prec c : c \notin S)$.

(b) Element b of S is the *greatest element of* S if $b \in S \wedge (\forall c \mid c \in S : c \preceq b)$.

(c) Element b is an *upper bound of* S if $(\forall c \mid c \in S : c \preceq b)$. (An upper bound of S need not be in S.)

(d) Element b is the *least upper bound of* S, written $lub.S$, if b is an upper bound and if every upper bound c satisfies $b \preceq c$.

Relational databases.

(14.56.1) **Definition, select:** For Relation R and predicate F, which may contain names of fields of R, $\sigma(R, F) = \{t \mid t \in R \wedge F\}$

(14.56.2) **Definition, project:** For $A_1, \ldots, A_m$ a subset of the names of the fields of relation R, $\pi(R, A_1, \ldots, A_m) = \{t \mid t \in R : \langle t.A_1, t.A_2, \ldots, t.A_m \rangle\}$

(14.56.3) **Definition, natural join:** For Relations $R1$ and $R2$, $R1 \bowtie R2$ has all the attributes that $R1$ and $R2$ have, but if an attribute appears in both, then it appears only once in the result; further, only those tuples that agree on this common attribute are included.

Growth of Functions

(g.1) **Definition of asymptotic upper bound:** For a given function $g.n$, $O(g.n)$, pronounced "big-oh of g of n", is the set of functions
$\{f.n \mid (\exists c, n_0 \mid c > 0 \wedge n_0 > 0 : (\forall n \mid n \geq n_0 : 0 \leq f.n \leq c \cdot g.n)\ )\}$

(g.2) O**-notation:** $f.n = O(g.n)$ means function $f.n$ is in the set $O(g.n)$.

(g.3) **Definition of asymptotic lower bound:** For a given function $g.n$, $\Omega(g.n)$, pronounced "big-omega of g of n", is the set of functions
$\{f.n \mid (\exists c, n_0 \mid c > 0 \wedge n_0 > 0 : (\forall n \mid n \geq n_0 : 0 \leq c \cdot g.n \leq f.n)\ )\}$

(g.4) Ω**-notation:** $f.n = \Omega(g.n)$ means function $f.n$ is in the set $\Omega(g.n)$.

(g.5) **Definition of asymptotic tight bound:** For a given function $g.n$, $\Theta(g.n)$, pronounced "big-theta of g of n", is the set of functions
$\{f.n \mid (\exists c_1, c_2, n_0 \mid c_1 > 0 \wedge c_2 > 0 \wedge n_0 > 0 :$
$(\forall n \mid n \geq n_0 : 0 \ \leq\ c_1 \cdot g.n \ \leq\ f.n \ \leq\ c_2 \cdot g.n)\ )\}$

(g.6) Θ**-notation:** $f.n = \Theta(g.n)$ means function $f.n$ is in the set $\Theta(g.n)$.

(g.7) $f.n = \Theta(g.n)$ if and only if $f.n = O(g.n)$ and $f.n = \Omega(g.n)$

Comparison of functions.

(g.8) **Reflexivity:**

(a) $f.n = O(f.n)$

(b) $f.n = \Omega(f.n)$

(c) $f.n = \Theta(f.n)$

(g.9) **Symmetry:** $f.n = \Theta(g.n) \equiv g.n = \Theta(f.n)$

(g.10) **Transpose symmetry:** $f.n = O(g.n) \equiv g.n = \Omega(f.n)$

(g.11) **Transitivity:**

(a) $f.n = O(g.n) \wedge g.n = O(h.n) \Rightarrow f.n = O(h.n)$

(b) $f.n = \Omega(g.n) \wedge g.n = \Omega(h.n) \Rightarrow f.n = \Omega(h.n)$

(c) $f.n = \Theta(g.n) \wedge g.n = \Theta(h.n) \Rightarrow f.n = \Theta(h.n)$

(g.12) Define an *asymptotically positive polynomial* $p.n$ *of degree* d to be $p.n = (\Sigma i \mid 0 \le i \le d : a_i n^i)$ where the constants $a_0, a_1, \ldots, a_d$ are the *coefficients* of the polynomial and $a_d > 0$. Then $p.n = \Theta(n^d)$.

(g.13) (a) $O(1) \subset O(\lg n) \subset O(n) \subset O(n \lg n) \subset O(n^2) \subset O(n^3) \subset O(2^n)$

(b) $\Omega(1) \supset \Omega(\lg n) \supset \Omega(n) \supset \Omega(n \lg n) \supset \Omega(n^2) \supset \Omega(n^3) \supset \Omega(2^n)$

A Theory of Integers

Minimum and maximum.

(15.53) **Definition of $\downarrow$:** $(\forall z \mid : z \le x \downarrow y \equiv z \le x \wedge z \le y)$

Definition of $\uparrow$: $(\forall z \mid : z \ge x \uparrow y \equiv z \ge x \wedge z \ge y)$

(15.54) **Symmetry:**

(a) $x \downarrow y = y \downarrow x$ (b) $x \uparrow y = y \uparrow x$

(15.55) **Associativity:**

(a) $(x \downarrow y) \downarrow z = x \downarrow (y \downarrow z)$ (b) $(x \uparrow y) \uparrow z = x \uparrow (y \uparrow z)$

Restrictions. Although $\downarrow$ and $\uparrow$ are symmetric and associative, they do not have identities over the integers. Therefore, axiom (8.13) empty range does not apply to $\downarrow$ or $\uparrow$. Also, when using range-split axioms, no range should be *false*.

(15.56) **Idempotency:**

(a) $x \downarrow x = x$ (b) $x \uparrow x = x$

Divisibility.

(15.77) **Definition of $\mid$:** $c \mid b \equiv (\exists d \mid : c \cdot d = b)$

(15.78) $c \mid c$

(15.79) $c \mid 0$

(15.80) $1 \mid b$

(15.80.1) $-b \mid c \equiv b \mid c$

(15.80.2) $-1 \mid b$

(15.81) $c \mid 1 \;\Rightarrow\; c = 1 \vee c = -1$

(15.81.1) $c \mid 1 \;\equiv\; c = 1 \vee c = -1$

(15.82) $d \mid c \wedge c \mid b \;\Rightarrow\; d \mid b$

(15.83) $b \mid c \wedge c \mid b \;\equiv\; b = c \vee b = -c$

(15.84) $b \mid c \;\Rightarrow\; b \mid c \cdot d$

(15.85) $b \mid c \;\Rightarrow\; b \cdot d \mid c \cdot d$

(15.86) $1 < b \wedge b \mid c \;\Rightarrow\; \neg(b \mid (c+1))$

(15.87) **Theorem:** Given integers b, c with $c > 0$, there exist (unique) integers q and r such that $b = q \cdot c + r$, where $0 \leq r < c$.

(15.89) **Corollary:** For given b, c, the values q and r of Theorem (15.87) are unique.

Greatest common divisor.

(15.90) **Definition of $\div$ and mod for operands b and c, $c \neq 0$:**
$b \div c = q,\; b \,\mathbf{mod}\, c = r$ where $b = q \cdot c + r$ and $0 \leq r < c$

(15.91) $b = c \cdot (b \div c) + b \,\mathbf{mod}\, c$ for $c \neq 0$

(15.92) **Definition of gcd:**
$b \,\mathbf{gcd}\, c = (\uparrow d \mid d \mid b \wedge d \mid c : d)$ for b, c not both 0
$0 \,\mathbf{gcd}\, 0 = 0$

(15.94) **Definition of lcm :**
$b \,\mathbf{lcm}\, c = (\downarrow k{:}\, \mathbb{Z}^{+} \mid b \mid k \wedge c \mid k : k)$ for $b \neq 0$ and $c \neq 0$
$b \,\mathbf{lcm}\, c = 0$ for $b = 0$ or $c = 0$

Properties of gcd.

(15.96) **Symmetry:** $b \,\mathbf{gcd}\, c = c \,\mathbf{gcd}\, b$

(15.97) **Associativity:** $(b \,\mathbf{gcd}\, c) \,\mathbf{gcd}\, d = b \,\mathbf{gcd}\, (c \,\mathbf{gcd}\, d)$

(15.98) **Idempotency:** $(b \,\mathbf{gcd}\, b) = abs.b$

(15.99) **Zero:** $1 \,\mathbf{gcd}\, b = 1$

(15.100) **Identity:** $0 \,\mathbf{gcd}\, b = abs.b$

(15.101) $b \,\mathbf{gcd}\, c = (abs.b) \,\mathbf{gcd}\, (abs.c)$

(15.102) $b \,\mathbf{gcd}\, c = b \,\mathbf{gcd}\, (b + c) = b \,\mathbf{gcd}\, (b - c)$

(15.103) $b = a \cdot c + d \;\Rightarrow\; b \,\mathbf{gcd}\, c = c \,\mathbf{gcd}\, d$

(15.104) **Distributivity:** $d \cdot (b \,\mathbf{gcd}\, c) = (d \cdot b) \,\mathbf{gcd}\, (d \cdot c)$ for $0 \leq d$

(15.105) **Definition of relatively prime $\perp$:** $b \perp c \;\equiv\; b \,\mathbf{gcd}\, c = 1$

(15.107) **Inductive definition of gcd:**
$b \,\mathbf{gcd}\, 0 = b$
$b \,\mathbf{gcd}\, c = c \,\mathbf{gcd}\, (b \,\mathbf{mod}\, c)$

(15.108) $(\exists x, y \mid : x \cdot b + y \cdot c = b \,\mathbf{gcd}\, c)$ for all $b, c{:}\, \mathbb{N}$

(15.111) $k \mid b \;\wedge\; k \mid c \;\equiv\; k \mid (b \,\mathbf{gcd}\, c)$

COMBINATORIAL ANALYSIS

(16.1) **Rule of sum:** The size of the union of n (finite) pairwise disjoint sets is the sum of their sizes.

(16.2) **Rule of product:** The size of the cross product of n sets is the product of their sizes.

(16.3) **Rule of difference:** The size of a set with a subset of it removed is the size of the set minus the size of the subset.

(16.4) **Definition:** $P(n, r) = n!/(n - r)!$

(16.5) The number of r-permutations of a set of size n equals $P(n, r)$.

(16.6) The number of r-permutations with repetition of a set of size n is n^r.

(16.7) The number of permutations of a bag of size n with k distinct elements occurring $n_1, n_2, \ldots, n_k$ times is $\dfrac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$.

(16.9) **Definition:** The *binomial coefficient* $\binom{n}{r}$, which is read as "n choose r", is defined by $\dbinom{n}{r} = \dfrac{n!}{r! \cdot (n - r)!}$ for $0 \leq r \leq n$.

(16.10) The number of r-combinations of n elements is $\binom{n}{r}$.

(16.11) The number $\binom{n}{r}$ of r-combinations of a set of size n equals the number of permutations of a bag that contains r copies of one object and $n - r$ copies of another.

A THEORY OF GRAPHS

(19.1) **Definition:** Let V be a finite, nonempty set and E a binary relation on V. Then $G = \langle V, E \rangle$ is called a *directed graph*, or *digraph*. An element of V is called a *vertex*; an element of E is called an *edge*.

(19.1.1) **Definitions:**

(a) In an *undirected* graph $\langle V, E \rangle$, E is a set of *unordered* pairs.
(b) In a *multigraph* $\langle V, E \rangle$, E is a *bag* of undirected edges.
(c) The *indegree* of a vertex of a digraph is the number of edges for which it is an end vertex.
(d) The *outdegree* of a vertex of a digraph is the number of edges for which it is a start vertex.
(e) The *degree* of a vertex is the sum of its indegree and outdegree.
(f) An edge $\langle b, b \rangle$ for some vertex b is a *self-loop*.
(g) A digraph with no self-loops is called *loop-free*.

(19.3) The sum of the degrees of the vertices of a digraph or multigraph equals $2 \cdot \#E$.

(19.4) In a digraph or multigraph, the number of vertices of odd degree is even.

(19.4.1) **Definition:** A *path* has the following properties.

(a) A path starts with a vertex, ends with a vertex, and alternates between vertices and edges.

(b) Each directed edge in a path is preceded by its start vertex and followed by its end vertex. An undirected edge is preceded by one of its vertices and followed by the other.

(c) No edge appears more than once.

(19.4.2) **Definitions:**

(a) A *simple* path is a path in which no vertex appears more than once, except that the first and last vertices may be the same.

(b) A *cycle* is a path with at least one edge, and with the first and last vertices the same.

(c) An undirected multigraph is *connected* if there is a path between any two vertices.

(d) A digraph is *connected* if making its edges undirected results in a connected multigraph.

(19.6) If a graph has a path from vertex b to vertex c, then it has a simple path from b to c.

(19.6.1) **Definitions:**

(a) An *Euler path* of a multigraph is a path that contains each edge of the graph exactly once.

(b) An *Euler circuit* is an Euler path whose first and last vertices are the same.

(19.8) An undirected connected multigraph has an Euler circuit iff every vertex has even degree.

(19.8.1) **Definitions:**

(a) A *complete graph* with n vertices, denoted by K_n, is an undirected, loop-free graph in which there is an edge between every pair of distinct vertices.

(b) A *bipartite graph* is an undirected graph in which the set of vertices are partitioned into two sets X and Y such that each edge is incident on one vertex in X and one vertex in Y.

(19.10) A path of a bipartate graph is of even length iff its ends are in the same partition element.

(19.11) A connected graph is bipartate iff every cycle has even length.

(19.11.1) **Definition:** A *complete bipartate graph* $K_{m,n}$ is a bipartite graph in which one partition element X has m vertices, the other partition element Y has n vertices, and there is an edge between each vertex of X and each vertex of Y.

(19.11.2) **Definitions:**

(a) A *Hamilton path* of a graph or digraph is a path that contains each vertex exactly once, except that the end vertices of the path may be the same.

(b) A *Hamilton circuit* is a Hamilton path that is a cycle.

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